MACMILLAN'S CANADIAN SCHOOL SERIES

JUNIOR ALGEBRA

HALL & KNIGHT.

JUNIOR ALGEBRA
FOR SCHOOLS

CONTAINING A FULL TREATMENT OF GRAPHS

WITH ANSWERS

BY

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PREFACE

This edition, called Junior Algebra, is a modification of the new and enlarged eighth edition of 1907, of Elementary Algebra, by Messrs. Hall and Knight, and has been specially adapted for Canadian Schools. One feature of the present edition is the large number of easy examples on the Theory of Quadratic Equations.

In view of the importance of the Graphic Method a chapter giving a very simple and continuous treatment of the subject has been placed at the end of the book. Teachers who wish to introduce graphs at an earlier stage will find suggestions at the beginning of Chapter XXXV., as to the best way to proceed. In graphical work it is most important that squared paper should be used, good in quality and accurately ruled to inches and tenths of an inch. Experience shows that anything on a smaller scale (such as "millimetre paper") is practically worthless in the hands of beginners.

As the subject of ratio and proportion is begun at an earlier stage in Geometry than formerly, it has been thought wise to give a brief treatment of this subject so that the Algebraic method may supplement the Geometrical. All names such as "duplicate," "subduplicate," etc., have been excluded. As ratio is closely related to fractions, chapter XXXIV. could be read, if thought desirable, immediately after the study of fractions.

SUGGESTIONS FOR A FIRST COURSE.

In the first thirty chapters an asterisk has been placed before all articles and examples which may conveniently be omitted in a first course. Notes are occasionally given suggesting the most suitable place for a section which may have to be postponed.

For those who wish to defer to a later stage all the rules dependent on 'Long' Multiplication and Division, so as to reach Quadratic Equations earlier, the following detailed course is recommended.

Chap. I. Arts. 1-11, 13-15. [Omit Art. 12, Examples I. c.]

Chap. II.-V. Arts. 16-40. [Omit all the rest of Chap. V., *except Art.* 44.]

Chap. VI. Arts. 46-50. [Omit Arts. 51-55.]

Chaps. VII.-XIII. Arts. 56-107. In connection with Chap. XIII. Arts. 311-318 on Easy Graphs may be read.

Chaps. XIV., XV. Arts. 108-113. [Omit Arts. 114, 115.]

Chap. XVI. Arts. 116-118a. [Omit Arts. 119-124.]

Chap. XVII. Arts. 125-136. [Omit Arts. 136a-137.]

Chap. XVIII. Arts. 138, 139. [Omit Arts. 140-148.]

Chap. XIX. [Omit Arts. 152, 153.]

Chap. XX. [Omit Arts. 159-163.]

Chap. XXI. [Omit Arts. 171, 172.]

Chap. XXII. Arts. 173-179. [Omit Arts. 180-185.]

[Chaps. XXIII., XXIV. may be taken later.]

Chap. XXV. **Quadratic Equations.** In connection with this chapter Arts. 325-334 may be read.

From this point the omitted sections must be taken at the discretion of the Teacher.

CONTENTS.

ALGEBRA.

CHAPTER I.

DEFINITIONS. SUBSTITUTIONS.

1. ALGEBRA treats of quantities as in Arithmetic, but with greater generality; for while the quantities used in arithmetical processes are denoted by *figures* which have one single definite value, algebraical quantities are denoted by *symbols* which may have any value we choose to assign to them.

The symbols employed are letters, usually those of our own alphabet; and, though there is no restriction as to the numerical values a symbol may represent, it is understood that in the same piece of work it keeps the same value throughout. Thus, when we say "let $a=1$," we do not mean that a must have the value 1 always, but only in the particular example we are considering. Moreover, we may operate with symbols without assigning to them any particular numerical value at all; indeed it is with such operations that Algebra is chiefly concerned.

We begin with the definitions of Algebra, premising that the symbols $+$, $-$, $\times$, $\div$, $=$, will have the same meanings as in Arithmetic. Also, for the present it will be assumed that all the algebraical symbols employed denote integral numbers.

2. An **algebraical expression** is a collection of symbols; it may consist of one or more **terms**, which are separated from each other by the signs $+$ and $-$. Thus $7a+5b-3c-x+2y$ is an expression consisting of five terms.

Note. When no sign precedes a term the sign $+$ is understood.

3. **Expressions** are either **simple** or **compound.** A *simple expression* consists of *one* term, as $5a$. A *compound expression* consists of *two or more* terms. Compound expressions may be further distinguished. Thus an expression of *two* terms, as $3a-2b$, is called a **binomial** expression; one of *three* terms, as $2a-3b+c$, a **trinomial**; one of *more than three* terms a **multinomial.** Simple expressions are also spoken of as **monomials.**

4. When two or more quantities are multiplied together the result is called the **product.** One important difference between the notation of Arithmetic and Algebra should be here remarked. In Arithmetic the product of 2 and 3 is written 2×3, whereas in Algebra the product of a and b may be written in any of the forms $a \times b$, $a \,.\, b$, or ab. The form ab is the most usual. Thus, if $a=2$, $b=3$, the product $ab=a \times b=2 \times 3=6$; but in Arithmetic 23 means "twenty-three," or $2 \times 10+3$.

5. Each of the quantities multiplied together to form a product is called a **factor** of the product. Thus 5, a, b, are the factors of the product $5ab$.

6. When one of the factors of an expression is a numerical quantity, it is called the **coefficient** of the remaining factors. Thus, in the expression $5ab$, 5 is the coefficient. But the word coefficient is also used in a wider sense, and it is sometimes convenient to consider any factor, or factors, of a product as the coefficient of the remaining factors. Thus, in the product $6abc$, $6a$ may be appropriately called the coefficient of bc. A coefficient which is not merely numerical is sometimes called a **literal coefficient.**

Note. When the coefficient is unity it is usually omitted. Thus we do not write $1a$, but simply a.

7. If a quantity be multiplied by itself any number of times, the product is called a **power** of that quantity, and is expressed by writing the number of factors to the right of the quantity and above it. Thus

$a \times a$ is called the *second power* of a, and is written a^2;
$a \times a \times a$...............*third power* of a, ... a^3;

and so on.

The number which expresses the power of any quantity is called its **index** or **exponent.** Thus 2, 5, 7 are respectively the indices of a^2, a^5, a^7.

Note. a^2 is usually read "a squared"; a^3 is read "a cubed"; a^4 is read "a to the fourth"; and so on.

When the index is unity it is omitted. Thus we do not write a^1, but simply a. Thus a, $1a$, a^1, $1a^1$ all have the same meaning.

8. The beginner must be careful to distinguish between *coefficient* and *index*.

Example 1. What is the difference in meaning between $3a$ and a^3?

By $3a$ we mean the product of the quantities 3 and a.

By a^3 we mean the third power of a; that is, the product of the quantities a, a, a.

Thus, if $a=4$,

$$3a=3\times a=3\times 4=12;$$
$$a^3=a\times a\times a=4\times 4\times 4=64.$$

Example 2. If $b=5$, distinguish between $4b^2$ and $2b^4$.

Here $4b^2=4\times b\times b=4\times 5\times 5=100$;

whereas $2b^4=2\times b\times b\times b\times b=2\times 5\times 5\times 5\times 5=1250$.

Example 3. If $a=4$, $x=1$, find the value of $5x^a$.

Here $5x^a=5\times x\times x\times x\times x=5\times 1\times 1\times 1\times 1=5$.

Note. The beginner should observe that every power of 1 is 1.

9. In arithmetical multiplication the order in which the factors of a product are written is immaterial. For instance 3×4 means 4 sets of 3 units, and 4×3 means 3 sets of 4 units; in each case we have 12 units in all. Thus

$$3\times 4=4\times 3.$$

In a similar way,

$$3\times 4\times 5=4\times 3\times 5=4\times 5\times 3;$$

and it is easy to see that the same principle holds for the product of any number of arithmetical quantities.

In like manner in Algebra ab and ba each denote the product of the two quantities represented by the letters a and b, and have therefore the same value. Again, the expressions abc, acb, bac, bca, cab, cba have the same value, each denoting the product of the three quantities a, b, c. It is immaterial in what order the factors of a product are written; it is usual, however, to arrange them in alphabetical order.

Fractional coefficients which are greater than unity are usually kept in the form of improper fractions.

Example. If $a=6$, $x=7$, $z=5$, find the value of $\frac{13}{10}axz$.

Here $\frac{13}{10}axz=\frac{13}{10}\times 6\times 7\times 5=273$.

EXAMPLES I. a.

If $a=7$, $b=2$, $c=1$, $x=5$, $y=3$, find the value of

1. $14x$.	2. x^3.	3. $3ax$.	4. a^3.	5. $5by$.
6. b^5.	7. $3b^2$.	8. $2xa$.	9. $6c^4$.	10. $4y^3$.
11. $7c^5$.	12. $9b^4$.	13. $8bcy$.	14. $7y^3$.	15. $8x^2$.

If $a=8$, $b=5$, $c=4$, $x=1$, $y=3$, find the value of

16. $9xy$.	17. $8b^3$.	18. $3x^5$.	19. x^8.	20. $7y^4$.
21. c^x.	22. b^y.	23. y^c.	24. x^b.	25. y^b.
26. a^y.	27. b^x.	28. a^c.	29. c^y.	30. $6bxy$

If $a=5$, $b=1$, $c=6$, $x=4$, find the value of

31. $\frac{3}{8}x^3$.	32. $\frac{1}{10}ax$.	33. 3^x.	34. 2^c.	35. 8^b.
36. 7^x.	37. $\frac{7}{15}acx$.	38. $\frac{1}{8}bcx$.	39. $\frac{2}{9}c^3$.	40. $\frac{x^5}{64}$.

10. When several different quantities are multiplied together a notation similar to that of Art. 7 is adopted. Thus $aabbbbcddd$ is written $a^2b^4cd^3$. And conversely $7a^3cd^2$ has the same meaning as $7\times a\times a\times a\times c\times d\times d$.

Example 1. If $x=5$, $y=3$, find the value of $4x^2y^3$.

$$\begin{aligned}4x^2y^3&=4\times 5^2\times 3^3\\&=4\times 25\times 27\\&=2700.\end{aligned}$$

Example 2. If $a=4$, $b=9$, $x=6$, find the value of $\frac{8bx^2}{27a^3}$.

$$\begin{aligned}\frac{8bx^2}{27a^3}&=\frac{8\times 9\times 6^2}{27\times 4^3}=\frac{8\times 9\times 36}{27\times 64}\\&=\frac{3}{2}=1\tfrac{1}{2}.\end{aligned}$$

11. If one factor of a product is equal to 0, the whole product must be equal to 0, *whatever values the other factors may have.* A factor 0 is usually called a **zero factor.**

For instance, if $x=0$ then ab^3xy^2 contains a zero factor. Therefore $ab^3xy^2=0$ when $x=0$, whatever be the values of a, b, y.

Again, if $c=0$, then $c^3=0$; therefore $ab^2c^3=0$, whatever values a and b may have.

Note. Every power of 0 is 0

EXAMPLES I. b.

If $a=7$, $b=2$, $c=0$, $x=5$, $y=3$, find the value of

1. $4ax^2$.	2. a^3b.	3. $8b^2y$.	4. $3xy^2$.
5. $\frac{3}{4}b^2x$.	6. $\frac{5}{6}b^3y^2$.	7. $\frac{2}{5}xy^4$.	8. a^3c.
9. a^2cy.	10. $8x^3y$.	11. $\frac{7}{20}ab^3x$.	12. $\frac{1}{9}x^2y^4$.

If $a=2$, $b=3$, $c=1$, $p=0$, $q=4$, $r=6$, find the value of

13. $\dfrac{3a^2r}{8b}$.	14. $\dfrac{8ab^2}{9q^2}$.	15. $\dfrac{6a^3c}{b^2}$.	16. $\dfrac{4cr^2}{9a^3}$.
17. $3a^2b^c$.	18. $\frac{5}{6}ba^r$.	19. $\dfrac{8b^q}{9a^r}$.	20. $5a^bc^r$.
21. $\dfrac{2a^2p}{7r}$.	22. 3^a2^b.	23. 2^ra^5.	24. c^bb^q.
25. $\dfrac{5a^rb^q}{64r^a}$.	26. $\dfrac{27a^q}{32}$.	27. $\dfrac{64}{q^r}$.	28. $\dfrac{b^r}{r^b}$.

[*The articles and examples marked with an asterisk may be postponed and taken in connection with* Chap. XVI.]

*12. DEFINITION. The **square root** of any proposed expression is that quantity whose square, or second power, is equal to the given expression. Thus the square root of 81 is 9, because $9^2=81$.

The square root of a is denoted by $\sqrt[2]{a}$, or more simply $\sqrt{a}$.

Similarly the **cube, fourth, fifth,** etc., **root** of any expression is that quantity whose third, fourth, fifth, etc., power is equal to the given expression.

The roots are denoted by the symbols $\sqrt[3]{}$, $\sqrt[4]{}$, $\sqrt[5]{}$, etc.

Examples. $\sqrt[3]{27}=3$; because $3^3=27$. $\sqrt[5]{32}=2$; because $2^5=32$.

The symbol $\sqrt{}$ is sometimes called the **radical sign.**

Example 1. Find the value of $5\sqrt{(6a^3b^4c)}$, when $a=3$, $b=1$, $c=8$.

$$5\sqrt{(6a^3b^4c)}=5\times\sqrt{(6\times3^3\times1^4\times8)}=5\times\sqrt{(6\times27\times8)}$$
$$=5\times\sqrt{1296}=5\times36=180.$$

Example 2. Find the value of $\sqrt[3]{\left(\dfrac{ab^4}{8x^3}\right)}$, when $a=9$, $b=3$, $x=5$.

$$\sqrt[3]{\left(\frac{ab^4}{8x^3}\right)}=\sqrt[3]{\left(\frac{9\times3^4}{8\times5^3}\right)}=\sqrt[3]{\left(\frac{9\times81}{8\times125}\right)}$$
$$=\sqrt[3]{\left(\frac{9\times9\times9}{1000}\right)}=\frac{9}{10}.$$

*EXAMPLES I. c.

If $a=8$, $c=0$, $k=9$, $x=4$, $y=1$, find the value of

1. $\sqrt{(2a)}$.	2. $\sqrt{(kx)}$.	3. $\sqrt{(2ax)}$.
4. $\sqrt{(2ak^2)}$.	5. $\sqrt[3]{(3k)}$.	6. $\sqrt[3]{(ax^3)}$.
7. $\sqrt[3]{(8x^3y^3)}$.	8. $\sqrt[3]{(cy^5)}$.	9. $2x\sqrt{(2ay)}$.
10. $5y\sqrt{(4kx)}$.	11. $3c\sqrt{(kx)}$.	12. $2xy\sqrt{(4y^5)}$.
13. $\sqrt{\left(\frac{8x^3}{ak}\right)}$.	14. $\sqrt{\left(\frac{25a}{2k}\right)}$.	15. $\sqrt{\left(\frac{16x}{49y^3}\right)}$.
16. $\sqrt{\left(\frac{ca^2}{16k}\right)}$.	17. $\sqrt[3]{\left(\frac{3a}{k^2}\right)}$.	18. $\sqrt[3]{\left(\frac{ax^3}{27y^3}\right)}$.
19. $\sqrt[3]{\left(\frac{ca}{3k}\right)}$.	20. $\sqrt[3]{\left(\frac{a^2k^2}{3x^3}\right)}$.	21. $\sqrt{\left(\frac{kax^2}{18y^3}\right)}$.

13. In the case of expressions which contain more than one term, each term can be dealt with singly by the rules already given, and by combining the terms the numerical value of the whole expression is obtained. When brackets ( ) are used, they will have the same meaning as in Arithmetic, indicating that the terms enclosed within them are to be considered as one quantity.

Example 1. When $c=5$, find the value of $c^4-4c+2c^3-3c^2$.

Here
$$c^4=5^4=5\times5\times5\times5=625;$$
$$4c=4\times5=20;$$
$$2c^3=2\times5^3=2\times5\times5\times5=250;$$
$$3c^2=3\times5^2=3\times5\times5=75.$$

Hence the value of the expression
$$=625-20+250-75=780.$$

Example 2. If $a=7$, $b=3$, $c=2$, find the value of
$$a(b+c)^2-c(a-b)^3.$$

The expression $=7(3+2)^2-2(7-3)^3=7\,.\,5^2-2\,.\,4^3=175-128=47.$

Example 3. When $a=5$, $b=3$, $c=1$, find the value of
$$a^2\cdot\frac{a-b}{b+2c}-b^2\cdot\frac{a-c}{(a+c)^2}.$$

The expression $=5^2\times\dfrac{5-3}{3+(2\times1)}-3^2\times\dfrac{5-1}{(5+1)^2}$
$$=25\times\frac{2}{5}-9\times\frac{4}{36}$$
$$=10-1=9.$$

14. By Art. 11 any term which contains a *zero factor* is itself zero, and may be called a *zero term.*

Example 1. If $a=2$, $b=0$, $x=5$, $y=3$, find the value of

$$5a^3 - ab^2 + 2x^2y + 3bxy.$$

The expression $=(5 \times 2^3) - 0 + (2 \times 5^2 \times 3) + 0$

$$=40+150=190.$$

Note. The two zero terms do not affect the result.

Example 2. Find the value of $\frac{3}{5}x^2 - a^2y + 7abx - \frac{5}{2}y^3$, when

$$a=5,\ b=0,\ x=7,\ y=1.$$

$$\frac{3}{5}x^2 - a^2y + 7abx - \frac{5}{2}y^3 = \frac{3}{5} \cdot 7^2 - 5^2 \cdot 1 + 0 - \frac{5}{2} \cdot 1^3$$
$$= 29\tfrac{2}{5} - 25 - 2\tfrac{1}{2} = 1\tfrac{9}{10}.$$

Example 3. Find the values of the expression $x^2 - 10x + 21$ when x has the values 0, 2, 3, 7, 8.

Here the following arrangement will be found convenient.

x	0	2	3	7	8
x^2	0	4	9	49	64
$10x$	0	20	30	70	80
$x^2 - 10x + 21$	21	5	0	0	5

Thus the required values are 21, 5, 0, 0, and 5.

15. In working examples the student should pay attention to the following hints.

1. Too much importance cannot be attached to neatness of style and arrangement. The beginner should remember that neatness is in itself conducive to accuracy.

2. The sign = should never be used except to connect quantities which are equal. Beginners should be particularly careful not to employ the sign of equality in any vague and inexact sense.

3. Unless the expressions are very short the signs of equality in the steps of the work should be placed one under the other.

4. It should be clearly brought out how each step follows from the one before it; for this purpose it will sometimes be advisable to add short verbal explanations; the importance of this will be seen later.

EXAMPLES I. d.

If $a=2$, $b=3$, $c=1$, $d=0$, find the numerical value of

1. $6a+5b-8c+9d$.
2. $3a-4b+6c+5d$.
3. $5a+3c-2b+6d$.
4. $ab+bc+ca-da$.
5. $6ab-3cd+2da-5cb+2db$.
6. $abc+bcd+cda+dab$.
7. $3abc-2bcd+2cda-4dab$.
8. $2bc+3cd-4da+5ab$.
9. $3bcd+5cda-7dab+abc$.
10. $a^2+b^2+c^2+d^2$.
11. $2a^2+3b^3-4c^4$.
12. $a^4+b^4-c^4$.

If $a=1$, $b=2$, $c=3$, $d=0$, find the numerical value of

13. $a^3+b^3+c^3+d^3$.
14. $\frac{1}{2}bc^3-a^3-b^3-\frac{3}{4}ab^3c$.
15. $3abc-b^2c-6a^3$.
16. $2a^2+2b^2+2c^2+2d^2-2bc-2cd-2da-2ab$.
17. $c^3+\frac{4}{5}ad^4-3a^3+b^2d$.
18. $a^2+2b^2+2c^2+d^2+2ab+2bc+\frac{2}{7}cd$.
19. $2c^2+2a^2+2b^2-4cb+6abcd$.
20. $13a^2+\frac{11}{9}c^4+20ab-16ac-16bc$.
21. $6ab-\frac{4}{3}ac^2-2a+\frac{1}{8}b^4-3d+\frac{4}{9}c^3$.
22. $a^2-c^2+b^2-d^2+2ab-2cd$.
23. $2ab-\frac{3}{4}b^3+3ac-2c-d+\frac{4}{15}ad$.
24. $125b^4c-9d^5+3abc^2d$.

If $a=2$, $b=1$, $c=3$, $x=4$, $y=6$, $z=0$, find the value of

25. $c^2(y-x)-b^2(c-a)$.
26. $(2a-c)(x+2y-z)$.
27. $\frac{2}{3}(c^2-z^2)+\frac{3}{5}(y^2-x^2)$.
28. $\frac{4}{9}(cy-2c^2)+\frac{3}{7}(xy-bc)$.
29. $\dfrac{a^2}{b^2}+\dfrac{b^2}{a^2}-\dfrac{2y}{x^2}$.
30. $\dfrac{a^2}{b^2}\cdot c^2+\dfrac{a^2}{b^2}+c^2$.
31. $\dfrac{(a+y)^2}{(x-z)^3}-\dfrac{6(c^2-a)}{7(a^2+x)}$.
32. $\dfrac{a^2-b^2}{a^2b^2}-\dfrac{(a+b+z)^2}{(b+c-z)^2}$.
33. $\dfrac{(a+b)^2}{(y-c)^2}-\dfrac{a(y-z)}{c(x+z)}$.
34. $\dfrac{(a+b+c)^2}{c(y-z)}-\dfrac{4(c-a)^3}{3(a+y)}$.

EXAMPLES I. e.

1. When x has the values 0, 3, 6, 8, 10 find the values of $x^2-9x+20$.

2. Find the values of $3+2x+\frac{x^2}{4}$ when x has the values 0, 1, 2, 3, 4.

3. Shew that $y^2-15y+56$ is 0 if $y=7$, and also if $y=8$. What is its value when $y=10$?

4. Find the values of the expression $\frac{x^3}{100}+\frac{x^2}{10}+2x$ when x has the values 2, 6, 8, 10.

5. Shew by substituting 10 for a and 3 for b that the two expressions

$$4(a-b)+3(a+b), \quad 5(a+b)+2(a-3b)$$

are equal.

Test the equality also when $a=6$, $b=0$.

6. Shew that $x^3-6x^2+11x-6$ is 0 for each of the values $x=1$, 2, 3. What is its value when $x=10$?

7. Shew that the expression x^3-13x^2+44x is equal to 32 when $x=1$, 4, or 8.

8. Shew that x^3+10x is equal to $7x^2$ for each of the values $x=0$, 2, 5. Which of the expressions is the greater, and by how much, when $x=6$?

9. By substituting 3 for x and 2 for y shew that the expressions

$$6x^3+7x^2y-y^3 \quad \text{and} \quad (2x+y)(3x-y)(x+y)$$

are equal.

10. Find the value of $4x^2+4x-3$ when $x=2$, and when $x=\frac{1}{2}$.

11. When $x=5$, shew that $4x^2+4x-3$ is equal to $9(x+8)$.

12. Shew that $6x^3-11x^2+3x$ is equal to 0 when $x=\frac{1}{3}$, and when $x=\frac{3}{2}$. Find its value in the form of a decimal when $x=\frac{1}{10}$.

Examples for Revision. (Oral.)

1. What do you understand by 63 and by 6 . 3?

2. What is meant by $45xy$ and $4 . 5xy$? If $x=4$, $y=5$, give the arithmetical value of each.

3. Which is the greater 245 or 2 . 4 . 5, and by how much?

4. Give the product of t and u in three ways.

5. If 5 boys have p marbles each, express algebraically how many they have in all. If $p=25$ what is the number?

6. If x cakes are to be shared equally among 6 boys, express algebraically how many each will have. If $x=42$ what is the number?

7. If 54 books are divided equally among c boys, express each boy's share algebraically. What is the arithmetical value if $c=6$?

8. What is the difference between "twice 3" and "3 squared"?

9. Give the expression for "thrice d," also that for the "cube of d." Give the arithmetical values if $d=2$.

10. Distinguish between "four times x" and "x to the fourth." Give the respective values when $x=3$.

11. The quantity c is to be multiplied by the quantity x. How is this expressed? Give the product if $c=7$ and $x=3$.

12. If x factors, each equal to c, are to be multiplied together, express this algebraically. What is the value if $x=3$ and the factor $c=7$?

13. The quantities a, b, c are to be added together. Express this algebraically. What is the answer if $a=5$, $b=7$, $c=11$?

14. The quantity r is to be taken from the quantity s. Give the algebraical expression that denotes this. What is the answer if $r=27$ and $s=41$?

15. A boy starts playing with x marbles and wins y. Express the number he then has. If $x=25$ and $y=9$, what number has he?

16. The same boy plays with his increased number and loses z. Express the number he then has. If $z=17$, how many has he left?

17. A farmer takes f sheep to market and sells g of them. How many has he left? What is the remainder if $f=64$ and $g=48$?

18. Another farmer takes k sheep to market and returns with l of them. How many has he sold? If $k=75$, and $l=32$, what is the number he has sold?

19. Give the sum and product of the three quantities a, b, c; and if $a=5$, $b=7$, $c=6$, give the arithmetical value of each.

20. If I walk y miles per hour for y hours, what is the algebraical expression for the length of my walk? If $y=4$, what is the answer?

CHAPTER II.

Negative Quantities. Addition of Like Terms.

16. In the preceding examples the sum of the terms to be subtracted has never been greater than the sum of the terms to be added; that is to say, every operation has been capable of being worked by Arithmetic. But in an example that reduces to a result such as $4-9$ the subtraction cannot be arithmetically performed, yet as an algebraical result such an expression can be explained; and, moreover, a subtractive term may stand alone and its meaning be quite plain.

17. Algebraical quantities which are preceded by the sign $+$ are said to be **positive**; those to which the sign $-$ is prefixed are said to be **negative**. When no sign is prefixed the $+$ sign is to be understood. These signs are frequently used to denote a *quality* possessed by the quantities to which they are attached, as explained in the following illustrations:

(i.) Suppose a trader gains \$100 and then loses \$70, the result of his trading is a *gain* of \$30, that is $+\$100-\$70=+\$30$; and the $+\$30$ denotes that he is \$30 better off than when he began.

But if he had first gained \$70 and then lost \$70, the loss would exactly balance the gain, that is $+\$70-\$70=\$0$. Thus he would be in the same position as when he began.

If, however, he had first gained \$70 and then lost \$100, the result of his trading would be a *loss* of \$30, that is $+\$70-\$100=-\$30$, and the $-\$30$ denotes that he is \$30 worse off than when he began, or that he now has a *debt* of \$30.

Thus we see that the $-\$30$ denotes a quantity *equal in magnitude, but opposite in character* to the $+\$30$.

(ii.) Again, suppose a man to row 60 yards up a stream, and then to drift down with the current for 40 yards, his position relative to the starting point would be $+60$ yards -40 yards $=+20$ yards, the $+20$ yards denoting the distance he was *up* stream from his starting point.

If he had rowed 40 yards up stream and then drifted down 60 yards, his position relative to the starting point would be $+40$ yards -60 yards $= -20$ yards, the -20 yards denoting the distance he was *down* stream from his starting point.

Thus we see that -20 yards denotes a distance *equal in magnitude, but opposite in direction* to that denoted by $+20$ yards.

(iii.) On a Centigrade thermometer 15° C. means 15° *above* the freezing point, and $-15°$ C. denotes 15° *below* freezing point.

From the above examples it will be understood that $+5$, for example, will denote a quantity *greater* than 0 by 5 units, whereas -5 will denote a quantity that is *less* than 0 by 5 units, the two quantities being of the same *absolute value* but *of opposite character.*

EXAMPLES II. a.

1. A trader gains \$20, loses \$42, and then gains \$10. Express algebraically the result of his three transactions.

2. Two cricket counties play 16 matches; one wins 10 and loses 6, and the other wins 7 and loses 9. Express the two results, allowing a gain of one point for a win and a loss of one point for a defeat.

3. In the night a Centigrade thermometer falls to $-8°$, and in the day-time it rises to 12°. How many degrees are there between the readings?

4. A Centigrade thermometer rises to 9° in the day-time and falls 15° during the night; what is the night reading?

5. A snail climbs 6 feet vertically upwards from a given point on a wall, slips down 15 feet, and then climbs 6 feet upwards again. Express algebraically his final position from his starting point.

6. Two men each fire 20 shots at a mark and agree to register 4 points for every hit and to deduct 3 points for every miss. One hits the mark 12 times, the other 8 times. Express algebraically their separate scores.

7. Each of three football teams plays 20 matches during the season. The A team wins 9 and loses 5, the B team wins 6 and loses 8, and the C team wins 9 and loses 9, the other games being drawn. If one point be allowed for a win, and one point deducted for a loss, place the three teams in order of merit and give the expressions that denote the results of the season's play.

Addition of Like Terms.

18. DEFINITION. When terms do not differ, or when they differ only in their numerical coefficients, they are called **like**, otherwise they are called **unlike**. Thus $3a$, $7a$; $5a^2b$, $2a^2b$; $3a^3b^2$, $-4a^3b^2$ are pairs of like terms; and $4a$, $3b$; $7a^2$, $9a^2b$ are pairs of unlike terms.

The rules for adding like terms are

Rule I. *The sum of a number of like terms is a like term.*

Rule II. *If all the terms are positive, add the coefficients.*

Example. Find the value of $8a+5a$.

Here we have to increase 8 things by 5 like things, and the aggregate is 13 of such things;

for instance, $\quad$ 8 lbs. + 5 lbs. = 13 lbs.

Hence also, $$8a+5a=13a.$$

Similarly, $$8a+5a+a+2a+6a=22a.$$

Rule III. *If all the terms are negative, add the coefficients numerically and prefix the minus sign to the sum.*

Example. To find the sum of $-3x$, $-5x$, $-7x$, $-x$.

Here we have to express, as one subtractive quantity, the *sum*, or total, of four subtractive quantities of like character. To subtract in succession 3, 5, 7, 1 like things would have the same effect as to take away $3+5+7+1$, or 16, such things in one operation.

Thus the sum of $-3x$, $-5x$, $-7x$, $-x$ is $-16x$.

Rule IV. *If the terms are not all of the same sign, add together separately the coefficients of all the positive terms and the coefficients of all the negative terms; the difference of these two results, preceded by the sign of the greater, will give the coefficient of the sum required.*

Example 1. Find the sum of $17x$ and $-8x$.

A gain of 17 followed by a loss of 8 would give as a result a gain of 9, for the difference of 17 and 8 is 9, and the gain, or positive term, is the greater.

Thus the sum of $17x$ and $-8x=9x$.

Example 2. The sum of $-17x$ and $8x=-9x$.

Example 3. Find the sum of $8a$, $-9a$, $-a$, $3a$, $4a$, $-11a$, a.

The sum of the coefficients of the positive terms is 16.

,, ,, ,, negative ,, 21.

The difference of these is 5 and the sign of the greater is negative; hence the required sum is $-5a$.

When a number of quantities are connected together by the signs + and −, the value of the result is the same in whatever order the terms are taken.

For example, in a series of combined losses and gains, the result is the same in whatever order the gains and losses are taken.

We may, therefore, add or subtract the terms in the most convenient order, which is usually that stated in Rule IV. above. This process is called **collecting terms.**

19. When quantities are connected by the signs + and −, the resulting expression is called their **algebraical sum.**

Thus $11a-27a+13a=-3a$ states that the algebraical sum of $11a$, $-27a$, $13a$ is equal to $-3a$.

Note. The sum of two quantities numerically equal but with opposite signs is zero. Thus the sum of $5a$ and $-5a$ is 0.

EXAMPLES II. b.

Find the sum of

1. $5a$, $7a$, $11a$, a, $23a$.

2. $4x$, x, $3x$, $7x$, $9x$.

3. $7b$, $10b$, $11b$, $9b$, $2b$.

4. $6c$, $8c$, $2c$, $15c$, $19c$, $100c$, c.

5. $-3x$, $-5x$, $-11x$, $-7x$.

6. $-5b$, $-6b$, $-11b$, $-18b$.

7. $-3y$, $-7y$, $-y$, $-2y$, $-4y$.

8. $-c$, $-2c$, $-50c$, $-13c$.

9. $-11b$, $-5b$, $-3b$, $-b$.

10. $5x$, $-x$, $-3x$, $2x$, $-x$.

11. $26y$, $-11y$, $-15y$, y, $-3y$, $2y$.

12. $5f$, $-9f$, $-3f$, $21f$, $-30f$.

13. $2s$, $-3s$, s, $-s$, $-5s$, $5s$.

14. $7y$, $-11y$, $16y$, $-3y$, $-2y$.

15. $5x$, $-7x$, $-2x$, $7x$, $2x$, $-5x$.

16. $7ab$, $-3ab$, $-5ab$, $2ab$, ab.

Find the value of

17. $-9x^2+11x^2+3x^2-4x^2$.

18. $3a^2x-18a^2x+a^2x$.

19. $3a^3-7a^3-8a^3+2a^3-11a^3$.

20. $4x^3-5x^3-8x^3-7x^3$.

21. $4a^2b^2-a^2b^2-7a^2b^2+5a^2b^2-a^2b^2$.

22. $-9x^4-4x^4-12x^4+13x^4-7x^4$.

23. $7abcd-11abcd-41abcd+2abcd$.

24. $\frac{1}{2}x-\frac{1}{3}x+x+\frac{2}{3}x$.

25. $\frac{3}{2}a+\frac{3}{5}a-\frac{1}{2}a$.

26. $-5b+\frac{1}{4}b-\frac{3}{2}b+2b-\frac{1}{2}b+\frac{7}{4}b$.

27. $-\frac{5}{3}x^2-2x^2-\frac{2}{3}x^2+x^2+\frac{1}{2}x^2+\frac{11}{6}x^2$.

28. $-ab-\frac{1}{2}ab-\frac{1}{3}ab-\frac{1}{4}ab-\frac{1}{6}ab+ab+\frac{5}{12}ab$.

29. $\frac{2}{3}x-\frac{3}{4}x+\frac{5}{6}x-2x+\frac{11}{6}x-\frac{1}{3}x+x$.

30. $-\frac{5}{3}x^2-\frac{3}{4}x^2-\frac{4}{3}x^2-\frac{1}{4}x^2-x^2$.

CHAPTER III.

SIMPLE BRACKETS. ADDITION.

20. WHEN a number of arithmetical quantities are connected together by the signs + and −, the value of the result is the same in whatever order the terms are taken. This also holds in the case of algebraical quantities.

Thus $a-b+c$ is equivalent to $a+c-b$, for in the first of the two expressions b is taken from a, and c added to the result ; in the second c is added to a, and b taken from the result. Similar reasoning applies to all algebraical expressions. Hence we may write the terms of an expression in any order we please.

Thus it appears that the expression $a-b$ may be written in the equivalent form $-b+a$.

To illustrate this we may suppose, as in Art. 17, that a represents a gain of a dollars, and $-b$ a loss of b dollars : it is clearly immaterial whether the gain precedes the loss or the loss precedes the gain.

21. Brackets ( ) are used to indicate that the terms enclosed within them are to be considered as one quantity. The full use of brackets will be considered in Chap. VII. ; here we shall deal only with the simpler cases.

$8+(13+5)$ means that 13 and 5 are to be added and their sum added to 8. It is clear that 13 and 5 may be added separately or together without altering the result.

Thus $$8+(13+5)=8+13+5=26.$$

Similarly $a+(b+c)$ means that the sum of b and c is to be added to a.

Thus $$a+(b+c)=a+b+c.$$

$8+(13-5)$ means that to 8 we are to add the excess of 13 over 5; now if we add 13 to 8 we have added 5 too much, and must therefore take 5 from the result.

Thus $$8+(13-5)=8+13-5=16.$$

Similarly $a+(b-c)$ means that to a we are to add b, diminished by c.

Thus $$a+(b-c)=a+b-c \quad\text{.......................}(1)$$

In like manner,

$$a+b-c+(d-e-f)=a+b-c+d-e-f \quad\text{............}(2)$$

Conversely,

$$a+b-c+d-e-f=a+b-c+(d-e-f) \quad\text{............}(3).$$

Again, $a-b+c=a+c-b$, [Art. 20.]
$=$ the sum of a and $c-b$,
$=$ the sum of a and $-b+c$. [Art. 20.]

therefore $a-b+c=a+(-b+c)$ ....(4).

By considering the results (1), (2), (3), (4) we are led to the following rule:

Rule. *When an expression within brackets is preceded by the sign* $+$, *the brackets can be removed without making any change in the expression.*

Conversely: *Any part of an expression may be enclosed within brackets and the sign* $+$ *prefixed, the sign of every term within the brackets remaining unaltered.*

Thus the expression $a-b+c-d+e$ may be written in any of the following ways,

$$a+(-b+c-d+e),$$
$$a-b+(c-d+e),$$
$$a-b+c+(-d+e).$$

22. The expression $a-(b+c)$ means that from a we are to take the sum of b and c. The result will be the same whether b and c are subtracted separately or in one sum. Thus

$$a-(b+c)=a-b-c.$$

Again, $a-(b-c)$ means that from a we are to subtract the excess of b over c. If from a we take b we get $a-b$; but by so doing we shall have taken away c too much, and must therefore add c to $a-b$. Thus

$$a-(b-c)=a-b+c.$$

In like manner,

$$a-b-(c-d-e)=a-b-c+d+e.$$

Accordingly the following rule may be enunciated:

Rule. *When an expression within brackets is preceded by the sign* $-$, *the brackets may be removed if the sign of every term within the brackets be changed.*

Conversely: *Any part of an expression may be enclosed within brackets and the sign* $-$ *prefixed, provided the sign of every term within the brackets be changed.*

Thus the expression $a-b+c+d-e$ may be written in any of the following ways,

$$a-(+b-c-d+e),$$
$$a-b-(-c-d+e),$$
$$a-b+c-(-d+e).$$

We have now established the following results.

I. *Additions and subtractions may be made in any order*

Thus $a+b-c+d-e-f=a-c+b+d-f-e$
$$=a-c-f+d+b-e.$$

This is known as the **Commutative Law for Addition and Subtraction.**

II. *The terms of an expression may be grouped in any manner.*

Thus $a+b-c+d-e-f=(a+b)-c+(d-e)-f$
$$=a+(b-c)+(d-e)-f=a+b-(c-d)-(e+f).$$

This is known as the **Associative Law for Addition and Subtraction.**

Addition of Unlike Terms.

23. When two or more *like* terms are to be added together we have seen that they may be collected and the result expressed as a *single* like term. If, however, the terms are *unlike* they cannot be collected; thus in finding the sum of two unlike quantities a and b, all that can be done is to connect them by the sign of addition and leave the result in the form $a+b$.

Also by the rules for removing brackets, $a+(-b)=a-b$; that is, the algebraic sum of a and $-b$ is written in the form $a-b$.

It will be observed that in Algebra the word *sum* is used in a wider sense than in Arithmetic. Thus, in the language of Arithmetic, $a-b$ signifies that b is to be subtracted from a, and bears that meaning only; but in Algebra it is also taken to mean the sum of the two quantities a and $-b$ without any regard to the relative magnitudes of a and b.

Example 1. Find the sum of $3a-5b+2c$; $2a+3b-d$; $-4a+2b$.

$$\begin{aligned}\text{The sum } &= (3a-5b+2c)+(2a+3b-d)+(-4a+2b)\\ &= 3a-5b+2c+2a+3b-d-4a+2b)\\ &= 3a+2a-4a-5b+3b+2b+2c-d\\ &= a+2c-d,\end{aligned}$$

by collecting like terms.

The addition is, however, more conveniently effected by the following rule:

Rule. *Arrange the expressions in lines so that the like terms may be in the same vertical columns: then add each column beginning with that on the left.*

$$\begin{array}{rrrr} 3a & -5b & +2c & \\ 2a & +3b & & -d \\ -4a & +2b & & \\ \hline a & & +2c & -d \end{array}$$

The algebraical sum of the terms in the first column is a, that of the terms in the second column is zero. The single terms in the third and fourth columns are brought down without change.

Example 2. Add together $-5ab+6bc-7ac$; $8ab+3ac-2ad$; $-2ab+4ac+5ad$; $bc-3ab+4ad$.

$$\begin{array}{rrrr} -5ab & +6bc & -7ac & \\ 8ab & & +3ac & -2ad \\ -2ab & & +4ac & +5ad \\ -3ab & +\ bc & & +4ad \\ \hline -2ab & +7bc & & +7ad \end{array}$$

Here we first rearrange the expressions so that like terms are in the same vertical columns, and then add up each column separately.

EXAMPLES III. a.

Find the sum of

1. $a+2b-3c$; $-3a+b+2c$; $2a-3b+c$.
2. $3a+2b-c$; $-a+3b+2c$; $2a-b+3c$.
3. $-3x+2y+z$; $x-3y+2z$; $2x+y-3z$.
4. $-x+2y+3z$; $3x-y+2z$; $2x+3y-z$.
5. $4a+3b+5c$; $-2a+3b-8c$; $a-b+c$.
6. $-15a-19b-18c$; $14a+15b+8c$; $a+5b+9c$.
7. $25a-15b+c$; $13a-10b+4c$; $a+20b-c$.
8. $-16a-10b+5c$; $10a+5b+c$; $6a+5b-c$.
9. $5ax-7by+cz$; $ax+2by-cz$; $-3ax+2by+3cz$.
10. $20p+q-r$; $p-20q+r$; $p+q-20r$.

Add together the following expressions:

11. $-5ab+6bc-7ca$; $8ab-4bc+3ca$; $-2ab-2bc+4ca$.

12. $15ab-27bc-6ca$; $14ab-18bc+10ca$; $45bc-3ca-49ab$.

13. $5ab+bc-3ca$; $ab-bc+ca$; $-ab+2ca+bc$.

14. $pq+qr-rp$; $-pq+qr+rp$; $pq-qr+rp$.

15. $x+y+z$; $2x+3y-2z$; $3x-4y+z$.

16. $2a-3b+c$; $15a-21b-8c$; $24b+7c+3a$.

17. $4xy-9yz+2zx$; $-25xy+24yz-zx$; $23xy-15yz+zx$.

18. $17ab-13bc+8ca$; $-5ab+9bc-7ca$; $-7bc-ca+2ab$.

19. $47x-63y+z$; $-25x+15y-3z$; $-22x+15z+48y$.

20. $-17b-2c+23a$; $-9a+15b+7c$; $-13a+3b-4c$.

Dimension and Degree.
Ascending and Descending Powers.

24. Each of the letters composing a term is called a **dimension** of the term, and the number of letters involved is called the **degree** of the term. Thus the product abc is said to be *of three dimensions*, or *of the third degree*; and ax^4 is said to be *of five dimensions*, or *of the fifth degree.*

A numerical coefficient is not counted. Thus $8a^2b^5$ and a^2b^5 are each of *seven* dimensions.

The **degree of an expression** is the degree of the term of highest dimensions contained in it; thus a^4-8a^3+3a-5 is *an expression of the fourth degree*, and $a^2x-7b^2x^3$ is *an expression of the fifth degree.* But it is sometimes useful to speak of the dimensions of an expression with regard to some one of the letters it involves. For instance the expression ax^3-bx^2+cx-d is said to be of *three dimensions in x.*

A compound expression is said to be **homogeneous** when all its terms are of the same dimensions. Thus $8a^6-a^4b^2+9ab^5$ is a *homogeneous expression of six dimensions.*

25. Different powers of the same letter are unlike terms; thus the result of adding together $2x^3$ and $3x^2$ cannot be expressed by a single term, but must be left in the form $2x^3+3x^2$.

Similarly the algebraical sum of $5a^2b^2$, $-3ab^3$, and $-b^4$ is $5a^2b^2-3ab^3-b^4$. This expression is in its simplest form and cannot be abridged.

In adding together several algebraical expressions containing terms with different powers of the same letter, it will be found convenient to arrange all expressions in *descending* or *ascending* powers of that letter. This will be made clear by the following examples.

Example 1. Add together $3x^3+7+6x-5x^2$; $2x^2-8-9x$; $4x-2x^3+3x^2$; $3x^3-9x-x^2$; $x-x^2-x^3+4$.

$$\begin{array}{rrrr}
3x^3 & -5x^2 & +6x & +7 \\
 & 2x^2 & -9x & -8 \\
-2x^3 & +3x^2 & +4x & \\
3x^3 & -x^2 & -9x & \\
-x^3 & -x^2 & +x & +4 \\
\hline
3x^3 & -2x^2 & -7x & +3
\end{array}$$

In writing down the first expression we put in the first term the highest power of x, in the second term the next highest power, and so on till the last term, in which x does not appear. The other expressions are arranged in the same way, so that in each column we have *like powers of the same letter.*

Example 2. Add together $3ab^2-2b^3+a^3$; $5a^2b-ab^2-3a^3$; $8a^3+5b^3$; $9a^2b-2a^3+ab^2$.

$$\begin{array}{rrrr}
-2b^3 & +3ab^2 & & +a^3 \\
 & -ab^2 & +5a^2b & -3a^3 \\
5b^3 & & & +8a^3 \\
 & ab^2 & +9a^2b & -2a^3 \\
\hline
3b^3 & +3ab^2 & +14a^2b & +4a^3
\end{array}$$

Here each expression contains powers of two letters, and is arranged according to *descending* powers of b, and *ascending* powers of a.

EXAMPLES III. b.

Find the sum of

1. $2ab+3ca+6abc$; $-5ab+2bc-5abc$; $3ab-2bc-3ca$.

2. $2x^2-2xy+3y^2$; $4y^2+5xy-2x^2$; $x^2-2xy-6y^2$.

3. $3a^2-7ab-4b^2$; $-6a^2+9ab-3b^2$; $4a^2+ab+5b^2$.

4. x^2+xy-y^2; $-z^2+yz+y^2$; $-x^2+xz+z^2$.

5. $-x^2-3xy+3y^2$; $3x^2+4xy-5y^2$; x^2+xy+y^2.

6. x^3-x^2+x-1; $2x^2-2x+2$; $-3x^3+5x+1$.

7. $2x^3-x^2-x$; $4x^3+8x^2+7x$; $-6x^3-6x^2+x$.

8. $9x^2-7x+5$; $-14x^2+15x-6$; $20x^2-40x-17$.

9. $10x^3+5x+8$; $3x^3-4x^2-6$; $2x^3-2x-3$.

10. $a^3-ab+bc$; $ab+b^3-ca$; $ca-bc+c^3$.

11. $5a^3-3c^3+d^3$; $b^3-2a^3+3d^3$; $4c^3-2a^3-3d^3$.

Find the sum of

12. $6x^3-2x+1$; $2x^3+x+6$; x^2-7x^3+2x-4.

13. a^3-a^2+3a; $3a^3+4a^2+8a$; $5a^3-6a^2-11a$.

14. x^2+y^2-2xy; $2z^2-3y^2-4yz$; $2x^2-2z^2-3xz$.

15. x^3-2y^3+x; y^3-2x^3+y; $x^2+2y^2-x+y^3$.

16. $x^3+3x^2y+3xy^2$; $-3x^2y-6xy^2-x^3$; $3x^2y+4xy^2$.

17. $a^3+5ab^2+b^3$; $b^3-10ab^2-a^3$; $5ab^2-2b^3+2a^2b$.

18. $x^5-4x^4y-5x^3y^3$; $3x^4y+2x^3y^3-6xy^4$; $3x^3y^3+6xy^4-y^5$.

19. $a^3-4a^2b+6abc$; $a^2b-10abc+c^3$; b^3+3a^2b+abc.

20. $x^3-4x^2y+6xy^2$; $2x^2y-3xy^2+2y^3$; $y^3+3x^2y+4xy^2$.

Add together the following expressions:

21. $\frac{1}{2}a-\frac{1}{3}b$; $-a+\frac{2}{3}b$; $\frac{3}{4}a-b$.

22. $-\frac{1}{3}a-\frac{1}{4}b$; $-\frac{2}{3}a+\frac{3}{4}b$; $-2a-b$.

23. $-2a+\frac{5}{2}c$; $-\frac{1}{3}a-2b$; $\frac{8}{3}b-3c$.

24. $-\frac{13}{8}a-\frac{11}{4}c$; $2a-3b$; $\frac{11}{5}b-c$.

25. $\frac{2}{3}x^2+\frac{1}{3}xy-\frac{1}{4}y^2$; $-x^2-\frac{2}{3}xy+2y^2$; $\frac{2}{3}x^2-xy-\frac{5}{4}y^2$.

26. $3a^2-\frac{2}{5}ab-\frac{1}{2}b^2$; $-\frac{3}{2}a^2+2ab-\frac{2}{3}b^2$; $-\frac{2}{3}a^2-ab+b^2$.

27. $\frac{5}{8}x^2-\frac{1}{3}xy+\frac{3}{10}y^2$; $-\frac{3}{4}x^2+\frac{14}{15}xy-y^2$; $\frac{1}{2}x^2-xy+\frac{1}{5}y^2$.

28. $-\frac{3}{4}x^3+5ax^2-\frac{5}{8}a^2x$; $x^3-\frac{37}{8}ax^2+\frac{1}{2}a^2x$; $-\frac{1}{2}x^3+\frac{3}{4}a^2x$.

29. $\frac{3}{8}x^2-\frac{5}{3}xy-7y^2$; $\frac{2}{3}xy+\frac{18}{5}y^2$; $-\frac{5}{8}x^2+4y^2$.

30. $\frac{1}{2}a^3-2a^2b-\frac{3}{2}b^3$; $\frac{3}{2}a^2b-\frac{3}{4}ab^2+2b^3$; $-\frac{3}{2}a^3+ab^2+\frac{1}{2}b^3$.

CHAPTER IV.

SUBTRACTION.

26. THE simplest cases of Subtraction have already come under the head of addition of *like* terms, of which some are negative. [Art. 18.]

Thus
$$5a-3a=2a,$$
$$3a-7a=-4a,$$
$$-3a-6a=-9a.$$

Also, by the rule for removing brackets [Art. 22],
$$3a-(-8a)=3a+8a$$
$$=11a,$$
and
$$-3a-(-8a)=-3a+8a$$
$$=5a.$$

Subtraction of Unlike Terms.

27. The method is shewn in the following example.

Example. Subtract $3a-2b-c$ from $4a-3b+5c$.

The difference

$$=4a-3b+5c-(3a-2b-c)$$
$$=4a-3b+5c-3a+2b+c$$
$$=4a-3a-3b+2b+5c+c$$
$$=a-b+6c.$$

The expression to be subtracted is first enclosed in brackets with a minus sign prefixed, then on removal of the brackets the like terms are combined by the rules already explained in Art. 18.

It is, however, more convenient to arrange the work as follows, the signs of all the terms in the lower line being changed.

$$\begin{array}{r} 4a-3b+5c \\ -3a+2b+c \\ \hline a-b+6c \end{array}$$
by *addition*,

The like terms are written in the same vertical column, and each column is treated separately.

Rule. *Change the sign of every term in the expression to be subtracted, and add to the other expression.*

Note. It is not necessary that in the expression to be subtracted the signs should be *actually* changed; the operation of changing signs ought to be performed mentally.

Example 1. From $5x^2+xy$ take $2x^2+8xy-7y^2$.

$$\begin{array}{l} 5x^2+\ xy \\ \underline{2x^2+8xy-7y^2} \\ 3x^2-7xy+7y^2 \end{array}$$

In the first column we combine mentally $5x^2$ and $-2x^2$, the algebraic sum of which is $3x^2$. In the last column the sign of the term $-7y^2$ has to be changed before it is put down in the result.

Example 2. Subtract $3x^2-2x$ from $1-x^3$.

Terms containing different powers of the same letter being *unlike* must stand in different columns.

$$\begin{array}{rrrr} -x^3 & & & +1 \\ & 3x^2 & -2x & \\ \hline -x^3 & -3x^2 & +2x & +1 \end{array}$$

In the first and last columns, as there is nothing to be subtracted, the terms are put down without change of sign. In the second and third columns each sign has to be changed.

The re-arrangement of terms in the first line is not *necessary*, but it is convenient, because it gives the result of subtraction in descending powers of x.

EXAMPLES IV. a.

Subtract

1. $4a-3b+c$ from $2a-3b-c$.
2. $a-3b+5c$ from $4a-8b+c$.
3. $2x-8y+z$ from $15x+10y-18z$.
4. $15a-27b+8c$ from $10a+3b+4c$.
5. $-10x-14y+15z$ from $x-y-z$.
6. $-11ab+6cd$ from $-10bc+ab-4cd$.
7. $4a-3b+15c$ from $25a-16b-18c$.
8. $-16x-18y-15z$ from $-5x+8y+7z$.
9. $ab+cd-ac-bd$ from $ab+cd+ac+bd$.
10. $-ab+cd-ac+bd$ from $ab-cd+ac-bd$.

From

11. $3ab+5cd-4ac-6bd$ take $3ab+6cd-3ac-5bd$.
12. $yz-zx+xy$ take $-xy+yz-zx$.
13. $-2x^3-x^2-3x+2$ take x^3-x+1.
14. $-8x^2y+15xy^2+10xyz$ take $4x^2y-6xy^2-5xyz$.
15. $\frac{1}{2}a-b+\frac{1}{3}c$ take $\frac{1}{3}a+\frac{1}{2}b-\frac{1}{2}c$.
16. $\frac{3}{4}x+y-z$ take $\frac{1}{2}x-\frac{1}{2}y-\frac{1}{3}z$.
17. $-a-3b$ take $\frac{3}{2}a+\frac{1}{3}b-\frac{1}{2}c$.
18. $\frac{1}{2}x-\frac{3}{7}y+\frac{1}{10}z$ take $-\frac{1}{2}x+\frac{4}{7}y-\frac{1}{10}z$.
19. $-\frac{2}{3}x-\frac{3}{5}y-5z$ take $\frac{2}{3}x-\frac{3}{5}y-\frac{11}{3}z$.
20. $-\frac{1}{2}x+\frac{2}{3}y-\frac{1}{6}$ take $\frac{1}{3}x-\frac{3}{2}y-\frac{1}{6}$.

EXAMPLES IV. b.

From

1. $3xy - 5yz + 8zx$ take $-4xy + 2yz - 10zx$.
2. $-8x^2y^2 + 15x^3y + 13xy^3$ take $4x^2y^2 + 7x^3y - 8xy^3$.
3. $-8 + 6ab + a^2b^2$ take $4 - 3ab - 5a^2b^2$.
4. $a^2bc + b^2ca + c^2ab$ take $3a^2bc - 5b^2ca - 4c^2ab$.
5. $-7a^2b + 8ab^2 + cd$ take $5a^2b - 7ab^2 + 6cd$.
6. $-8x^2y + 5xy^2 - x^2y^2$ take $8x^2y - 5xy^2 + x^2y^2$.
7. $10a^2b^2 + 15ab^2 + 8a^2b$ take $-10a^2b^2 + 15ab^2 - 8a^2b$.
8. $4x^2 - 3x + 2$ take $-5x^2 + 6x - 7$.
9. $x^3 + 11x^2 + 4$ take $8x^2 - 5x - 3$.
10. $-8a^2x^2 + 5x^2 + 15$ take $9a^2x^2 - 8x^2 - 5$.

Subtract

11. $x^3 - x^2 + x + 1$ from $x^3 + x^2 - x + 1$.
12. $3xy^2 - 3x^2y + x^3 - y^3$ from $x^3 + 3x^2y + 3xy^2 + y^3$.
13. $b^3 + c^3 - 2abc$ from $a^3 + b^3 - 3abc$.
14. $7xy^2 - y^3 - 3x^2y + 5x^3$ from $8x^3 + 7x^2y - 3xy^2 - y^3$.
15. $x^4 + 5 + x - 3x^3$ from $5x^4 - 8x^3 - 2x^2 + 7$.
16. $a^3 + b^3 + c^3 - 3abc$ from $7abc - 3a^3 + 5b^3 - c^3$.
17. $1 - x + x^5 - x^4 - x^3$ from $x^4 - 1 + x - x^2$.
18. $7a^4 - 8a^2 + 3a^5 + a$ from $a^2 - 5a^3 - 7 + 7a^5$.
19. $10a^2b + 8ab^2 - 8a^3b^3 - b^4$ from $5a^2b - 6ab^2 - 7a^3b^3$.
20. $a^3 - b^3 + 8ab^2 - 7a^2b$ from $-8ab^2 + 15a^2b + b^3$.

From

21. $\frac{1}{2}x^2 - \frac{1}{3}xy - \frac{3}{2}y^2$ take $-\frac{3}{2}x^2 + xy - y^2$.
22. $\frac{2}{3}a^2 - \frac{5}{2}a - 1$ take $-\frac{2}{3}a^2 + a - \frac{1}{2}$.
23. $\frac{1}{3}x^2 - \frac{1}{2}x + \frac{1}{6}$ take $\frac{1}{3}x - 1 + \frac{1}{2}x^2$.
24. $\frac{3}{8}x^2 - \frac{2}{3}ax$ take $\frac{1}{3} - \frac{1}{4}x^2 - \frac{5}{6}ax$.
25. $\frac{3}{4}x^3 - \frac{1}{3}xy^2 - y^2$ take $\frac{1}{2}x^2y - \frac{5}{6}y^2 - \frac{1}{3}xy^2$.
26. $\frac{1}{8}a^3 - 2ax^2 - \frac{1}{3}a^2x$ take $\frac{1}{3}a^2x + \frac{1}{4}a^3 - \frac{3}{2}ax^2$.

MISCELLANEOUS EXAMPLES I.

1. Simplify (1) $4x - 2x^2 - (2x - 3x^2)$;
(2) $3a - 4b - (3b + a) - (5a - 8b)$.

2. To the sum of $2a - 3b - 2c$ and $2b - a + 7c$ add the sum of $a - 4c + 7b$ and $c - 6b$.

3. When $x = 3$, $y = 2$, $z = 0$, find the value of

(1) $x^2 + \frac{3}{2}y^3 - xyz^3$; (2) $\frac{1}{4}x^3y^4 + \frac{5z^2}{6}$.

4. Define *index, coefficient.* In the expressions $4x^2 + 3x$, $2x^3 + x^2$, $x^2 + 7x$, find (1) the sum of the indices, (2) the sum of the coefficients.

5. From $5x^3 + 3x - 1$ take the sum of

$$2x - 5 + 7x^2 \text{ and } 3x^2 + 4 - 2x^3 + x.$$

6. Subtract $3a - 7a^3 + 5a^2$ from the sum of

$$2 + 8a^2 - a^3 \text{ and } 2a^3 - 3a^2 + a - 2.$$

---

7. Distinguish between *like* and *unlike* terms. Pick out the like terms in the expression

$$a^3 - 3ab + b^2 - 2a^3 - a^2 + 3b^2 + 5ab + 7a^2.$$

8. Write down in as many ways as possible the result of adding together x, y, and z.

9. Subtract $5x^2 + 3x - 1$ from $2x^3$, and add the result to

$$3x^2 + 3x - 1.$$

10. If the number of dollars I possess is represented by $+a$, what will $-a$ denote?

11. Write down in algebraical symbols the result of diminishing $2a$ by the sum of $3b$ and $5c$.

12. When $x = 1$, $y = 2$, $z = 3$, find the value of the sum of $5x^2$, $-2x^3z$, $3y^4$. Also find the value of $2z^y - 3y^x$.

---

13. Add the sum of $2y - 3y^2$ and $1 - 5y^3$ to the remainder left when $1 - 2y^2 + y$ is subtracted from $5y^3$.

14. Explain clearly why $x - (y - z) = x - y + z$.

15. If $x = 4$, $y = 3$, $z = 2$, $a = 0$, find the value of

$$3x^2 - 2yz - ax + 5ax^2y.$$

16. Simplify $2a - b - (3a - 2b) + (2a - 3b) - (a - 2b)$.

17. Find the algebraical sum of the like terms in the expression

$$5a^3 - 4a^2b + b^3 + 6a^3b + 7ab^2 - 3a^2b + 4ab^3 + 8a^2b.$$

18. A boy works $x+y$ sums, of which only $y-2z$ are right; how many are wrong?

---

19. In the expression $3a^3 - 7a^2b + b^4$, point out the highest power, the lowest power, the positive terms, and the coefficient of a^2.

20. Take $x^2 - y^2$ from $3xy - 4y^2$, and add the remainder to the sum of $4xy - x^2 - 3y^2$ and $2x^2 + 6y^2$.

21. If $x=1$, $y=3$, $z=5$, $w=0$, find the value of

$$\sqrt{(3xy)} + \sqrt{(5xz)} + \sqrt{(3yw)}.$$

22. What is the *degree* of a term in an algebraical expression? In the expression $4x^6 - 3x^5a^2 + a^8$, what is the degree of the negative term?

23. Find the sum of $5a - 7b + c$ and $3b - 9a$, and subtract the result from $c - 4b$.

24. If $x=3$, $y=4$, $p=8$, $q=10$, find the value of

$$xyp + \frac{2y}{p-y} + 2q.$$

---

25. If x represents the date 10 A.D. what will $-3x$ stand for?

26. Add together $3x^2 - 7x + 5$ and $2x^3 + 5x - 3$, and diminish the result by $3x^2 + 2$.

27. In the expression

$$4a^2b^3 - b^4 + 3a^3b^2 + 5b^5 - ab^3x + 2x^3ab + abx^4 - a^2b^3,$$

point out which terms are *like*, and which are homogeneous. What is the degree of the expression?

28. Express in algebraical symbols the excess of the sum of a and b over c diminished by d.

29. A man walks $2a - b$ miles due North from a fixed point O, and then walks a distance $3a + 2b$ miles due South; what is his final position with regard to O?

30. What expression must be added to $5x^2 - 7x + 2$ to produce $7x^2 - 1$?

CHAPTER V.

MULTIPLICATION.

[*Part of this chapter may be taken at a later stage. See remark on page* 33. *The easy graphical work in* Arts. 304–315 *may be studied after* Examples v. b.]

28. MULTIPLICATION in its primary sense signifies repeated addition.

Thus $3 \times 4 = 3$ taken 4 times

$$= 3 + 3 + 3 + 3.$$

Here the multiplier contains four units, and the number of times we take 3 is the same as the number of units in 4.

Again $a \times b = a$ taken b times

$$= a + a + a + \ldots,$$

the number of terms being b.

Also $3 \times 4 = 4 \times 3$; and so long as a and b denote positive whole numbers, it is easy to show that $a \times b = b \times a$.

29. When the quantities to be multiplied together are not positive whole numbers, we may define multiplication as *an operation performed on one quantity which when performed on unity produces the other.* For example, to multiply $\frac{4}{5}$ by $\frac{3}{7}$, we perform on $\frac{4}{5}$ that operation which when performed on unity gives $\frac{3}{7}$; that is, we must divide $\frac{4}{5}$ into seven equal parts and take three of them. Now each part will be equal to $\frac{4}{5 \times 7}$, and the result of taking three of such parts is expressed by $\frac{4 \times 3}{5 \times 7}$.

Hence $$\frac{4}{5} \times \frac{3}{7} = \frac{4 \times 3}{5 \times 7}.$$

Also, by the last article,

$$\frac{4 \times 3}{5 \times 7} = \frac{3 \times 4}{7 \times 5} = \frac{3}{7} \times \frac{4}{5}$$

$$\therefore \frac{4}{5} \times \frac{3}{7} = \frac{3}{7} \times \frac{4}{5}.$$

The reasoning is clearly general, and we may now say that $a \times b = b \times a$, where a and b are any positive quantities, integral or fractional.

In the same way it easily follows that

$$\begin{aligned} abc &= a \times b \times c \\ &= (a \times b) \times c = (b \times a) \times c = bac \\ &= b \times (a \times c) = b \times c \times a = bca; \end{aligned}$$

that is, *the factors of a product may be taken in any order.* This is the **Commutative Law for Multiplication.**

Example. $2a \times 3b \times c = 2 \times 3 \times a \times b \times c = 6abc$.

30. Again, *the factors of a product may be grouped in any way we please.*

Thus $$\begin{aligned} abcd &= a \times b \times c \times d \\ &= (ab) \times (cd) = a \times (bc) \times d = a \times (bcd). \end{aligned}$$

This is the **Associative Law for Multiplication.**

31. Since, by definition, $a^3 = aaa$, and $a^5 = aaaaa$,

$$\therefore\ a^3 \times a^5 = aaa \times aaaaa = aaaaaaaa = a^8 = a^{3+5};$$

that is, *the index of a letter in a product is the sum of its indices in the factors of the product.* This is the **Index Law for Multiplication.**

Again, $5a^2 = 5aa$, and $7a^3 = 7aaa$.

$$\therefore\ 5a^2 \times 7a^3 = 5 \times 7 \times aaaaa = 35a^5.$$

When the expressions to be multiplied together contain powers of different letters, a similar method is used.

Example. $$\begin{aligned} 5a^3b^2 \times 8a^2bx^3 &= 5aaabb \times 8aabxxx \\ &= 40a^5b^3x^3. \end{aligned}$$

Note. The beginner must be careful to observe that in this process of multiplication *the indices of one letter cannot combine in any way with those of another.* Thus the expression $40a^5b^3x^3$ admits of no further simplification.

32. Rule. *To multiply two simple expressions together, multiply the coefficients together and prefix their product to the product of the different letters, giving to each letter an index equal to the sum of the indices that letter has in the separate factors.*

The rule may be extended to cases where more than two expressions are to be multiplied together.

Example 1. Find the product of x^2, x^3, and x^8.

The product $= x^2 \times x^3 \times x^8 = x^{2+3} \times x^8 = x^{2+3+8} = x^{13}$.

The product of three or more expressions is called **the continued product.**

Example 2. Find the continued product of $5x^2y^3$, $8y^2z^5$, and $3xz^4$.

The product $= 5x^2y^3 \times 8y^2z^5 \times 3xz^4 = 120x^3y^5z^9$.

Multiplication of a Compound Expression by a Simple Expression.

33. By definition,

$$\begin{aligned} (a+b)m &= m+m+m+\dots \text{ taken } a+b \text{ times} \\ &= (m+m+m+\dots \text{ taken } a \text{ times}), \end{aligned}$$

together with $(m+m+m+\dots$ taken b times$)$

$$= am + bm \quad\dots\dots(1).$$

Also $\quad (a-b)m = m+m+m+\dots$ taken $a-b$ times

$$= (m+m+m+\dots \text{ taken } a \text{ times}),$$

diminished by $(m+m+m+\dots$ taken b times$)$

$$= am - bm \quad\dots\dots(2).$$

Similarly $\quad (a-b+c)m = am - bm + cm.$

Thus it appears that *the product of a compound expression by a single factor is the algebraic sum of the partial products of each term of the compound expression by that factor.* This is known as the **Distributive Law for Multiplication.**

Note. It should be observed that for the present a, b, c denote positive whole numbers, and that a is supposed greater than b.

Examples. $\quad 3(2a+3b-4c) = 6a+9b-12c.$

$$(4x^2 - 7y - 8z^3) \times 3xy^2 = 12x^3y^2 - 21xy^3 - 24xy^2z^3.$$

EXAMPLES V. a.

Find the value of

1. $5x^2 \times 7x^5$. **2.** $4a^3 \times 5a^8$. **3.** $7ab \times 8a^3b^2$.

4. $6xy^2 \times 5x^3$. **5.** $8a^3b \times b^5$. **6.** $2abc \times 3ac^3$.

7. $2a^3b^3 \times 2a^3b^3$. **8.** $5a^2b \times 2a$. **9.** $4a^2b^3 \times 7a^5$.

10. $5a^4b^3 \times x^2y^2$. **11.** $x^3y^3 \times 6a^2x^4$. **12.** $abc \times xyz$.

13. $3a^4b^7x^3 \times 5a^3bx$. **14.** $4a^3bx \times 7b^2x^4$. **15.** $5a^2x \times 8cx$.

16. $5x^3y^3 \times 6a^3x^3$. **17.** $2x^2y \times x^5y^7$. **18.** $3a^3x^4y^7 \times a^2x^5y^9$.

Multiply together:

19. $ab+bc$ and a^3b.

20. $5ab-7bx$ and $4a^2bx^3$.

21. $5x+3y$ and $2x^2$.

22. $a^2+b^2-c^2$ and a^3b.

23. $bc+ca-ab$ and abc.

24. $5a^2+3b^2-2c^2$ and $4a^2bc^3$.

25. $5x^2y+xy^2-7x^2y^2$ and $3x^3$.

26. $6x^3-5x^2y+7xy^2$ and $8x^2y^3$.

27. $6a^3bc-7ab^2c^2$ and a^2b^2.

Multiplication of Compound Expressions.

34. If in Art. 33 we write $c+d$ for m in (1), we have

$$\begin{aligned}(a+b)(c+d)&=a(c+d)+b(c+d)\\&=(c+d)a+(c+d)b \qquad \text{[Art. 29.]}\\&=ac+ad+bc+bd \quad \ldots\ldots\ldots\ldots(3).\end{aligned}$$

Again, from (2)

$$\begin{aligned}(a-b)(c+d)&=a(c+d)-b(c+d)\\&=(c+d)a-(c+d)b\\&=ac+ad-(bc+bd)\\&=ac+ad-bc-bd \quad \ldots\ldots\ldots\ldots(4).\end{aligned}$$

Similarly, by writing $c-d$ for m in (1),

$$\begin{aligned}(a+b)(c-d)&=a(c-d)+b(c-d)\\&=(c-d)a+(c-d)b\\&=ac-ad+bc-bd \quad \ldots\ldots\ldots\ldots(5).\end{aligned}$$

Also, from (2)

$$\begin{aligned}(a-b)(c-d)&=a(c-d)-b(c-d)\\&=(c-d)a-(c-d)b\\&=ac-ad-(bc-bd)\\&=ac-ad-bc+bd \quad \ldots\ldots\ldots\ldots(6).\end{aligned}$$

If we consider each term on the right-hand side of (6), and the way in which it arises, we find that

$$\begin{aligned}(+a)\times(+c)&=+ac.\\(-b)\times(-d)&=+bd.\\(-b)\times(+c)&=-bc.\\(+a)\times(-d)&=-ad.\end{aligned}$$

These results enable us to state what is known as the **Rule of Signs** in multiplication.

Rule of Signs. *The product of two terms with like signs is positive; the product of two terms with unlike signs is negative.*

35. The rule of signs, and especially the use of the negative multiplier, will probably present some difficulty to the beginner. Perhaps the following numerical instances may be useful in illustrating the interpretation that may be given to multiplication by a negative quantity.

To multiply 3 by -4 we must do to 3 what is done to unity to obtain -4. Now -4 means that unity is taken 4 times and the result made negative; therefore $3\times(-4)$ implies that 3 is to be taken 4 times and the product made negative.

But 3 taken 4 times gives $+12$;

$$\therefore\ 3\times(-4)=-12.$$

Similarly -3×-4 indicates that -3 is to be taken 4 times, and the sign changed; the first operation gives -12, and the second $+12$.

Thus $$(-3)\times(-4)=+12.$$

Hence, *multiplication by a negative quantity indicates that we are to proceed just as if the multiplier were positive, and then change the sign of the product.*

Note on Arithmetical and Symbolical Algebra.

36. Arithmetical Algebra is that part of the science which deals solely with symbols and operations arithmetically intelligible. Starting from purely arithmetical definitions, we are enabled to prove certain fundamental laws.

Symbolical Algebra assumes these laws to be true in every case, and thence finds what meaning must be attached to symbols and operations which under unrestricted conditions no longer bear an arithmetical meaning. Thus the results of Arts. 33 and 34 were proved from arithmetical definitions which require the symbols to be positive whole numbers, such that $a>b$ and $c>d$. By the principles of symbolical Algebra we assume these results to be universally true when all restrictions are removed, and accept the interpretation to which we are led thereby.

Henceforth we are able to apply the Law of Distribution and the Rule of Signs without any restriction as to the symbols used. [See Art. 33, Note.]

37. To familiarize the beginner with the principles we have just explained we add a few examples in substitutions where some of the symbols denote negative quantities.

Example 1. If $a=-4$, find the value of a^3.

Here $a^3=(-4)^3=(-4)\times(-4)\times(-4)=-64$.

By repeated applications of the rule of signs it may easily be shewn that any *odd* power of a negative quantity is *negative*, and any *even* power of a negative quantity is positive.

Example 2. If $a=-1$, $b=3$, $c=-2$, find the value of $-3a^4bc^3$.

Here $-3a^4bc^3=-3\times(-1)^4\times3\times(-2)^3$
$=-3\times(+1)\times3\times(-8)$
$=72$.

We write down at once $(-1)^4=+1$, and $(-2)^3=-8$.

EXAMPLES V. b.

If $a=-2$, $b=3$, $c=-1$, $x=-5$, $y=4$, find the value of

1. $3a^2b$.	**2.** $8abc^2$.	**3.** $-5c^3$.	**4.** $6a^2c^2$.
5. $4c^3y$.	**6.** $3a^2c$.	**7.** $-b^2c^2$.	**8.** $3a^3c^2$.
9. $-7a^3bc$.	**10.** $-2a^4bx$.	**11.** $-4a^2c^4$.	**12.** $3c^3x^3$.
13. $5a^2x^2$.	**14.** $-7c^4xy$.	**15.** $-8ax^3$.	**16.** $4c^5x^3$.
17. $-5a^2b^2c^2$.	**18.** $-7a^3c^3$.	**19.** $8c^4x^3$.	**20.** $7a^5c^4$.

If $a=-4$, $b=-3$, $c=-1$, $f=0$, $x=4$, $y=1$, find the value of

21. $3a^2+bx-4cy$.

22. $2ab^2-3bc^2+2fx$.

23. $fa^2-2b^3-cx^3$.

24. $3a^2y^3-5b^2x-2c^3$.

25. $2a^3-3b^3+7cy^4$.

26. $3b^2y^4-4b^2f-6c^4x$.

27. $2\sqrt{(ac)}-3\sqrt{(xy)}+\sqrt{(b^2c^4)}$.

28. $3\sqrt{(acx)}-2\sqrt{(b^2y)}-6\sqrt{(c^2y)}$.

29. $7\sqrt{(a^2x)}-3\sqrt{(b^4c^2)}+5\sqrt{(f^2x)}$.

30. $3c\sqrt{(3bc)}-5\sqrt{(4c^2y^3)}-2cy\sqrt{(3bc^5)}$.

38. The following examples further illustrate the rule of signs and the law of indices.

Example 1. Multiply $4a$ by $-3b$.

By the rule of signs the product is negative; also $4a\times3b=12ab$;

$$\therefore\ 4a\times(-3b)=-12ab.$$

Example 2. Multiply $-5ab^3x$ by $-ab^3x$.

Here the absolute value of the product is $5a^2b^6x^2$, and by the rule of signs the product is positive;

$$\therefore\ (-5ab^3x)\times(-ab^3x)=5a^2b^6x^2.$$

Example 3. Find the continued product of $3a^2b$, $-2a^3b^2$, $-ab^4$.

$3a^2b\times(-2a^3b^2)=-6a^5b^3$;

$(-6a^5b^3)\times(-ab^4)=+6a^6b^7$.

Thus the complete product is $6a^6b^7$.

This result, however, may be written down at once: for

$$3a^2b\times2a^3b^2\times ab^4=6a^6b^7,$$

and by the rule of signs the required product is positive.

Example 4. Multiply $6a^3 - \frac{5}{3}a^2b - \frac{4}{5}ab^2$ by $-\frac{3}{4}ab^2$.

The product is the algebraical sum of the partial products formed according to the rule enunciated in Art. 37; thus

$$(6a^3 - \tfrac{5}{3}a^2b - \tfrac{4}{5}ab^2) \times (-\tfrac{3}{4}ab^2) = -\tfrac{9}{2}a^4b^2 + \tfrac{5}{4}a^3b^3 + \tfrac{3}{5}a^2b^4.$$

EXAMPLES V. c.

Multiply together:

1. ax and $-3ax$.
2. $-2abx$ and $-7abx$
3. a^2b and $-ab^2$.
4. $6x^2y$ and $-10xy$.
5. $-abcd$ and $-3a^2b^3c^4d^5$.
6. xyz and $-5x^2y^3z$.
7. $3xy+4yz$ and $-12xyz$.
8. $ab-bc$ and a^2bc^3.
9. $-x-y-z$ and $-3x$.
10. $a^2-b^2+c^2$ and abc.
11. $-ab+bc-ca$ and $-abc$.
12. $-2a^2b-4ab^2$ and $-7a^2b^2$.
13. $5x^2y-6xy^2+8x^2y^2$ and $3xy$.
14. $-7x^3y-5xy^3$ and $-8x^3y^3$.
15. $-5xy^2z+3xyz^2-8x^2yz$ and xyz.
16. $4x^2y^2z^2-8xyz$ and $-12x^3yz^3$.
17. $-13xy^2-15x^2y$ and $-7x^3y^3$.
18. $8xyz-10x^3yz^3$ and $-xyz$.
19. $abc-a^2bc-ab^2c$ and $-abc$.
20. $-a^2bc+b^2ca-c^2ab$ and $-ab$.

Find the product of

21. $2a-3b+4c$ and $-\frac{3}{2}a$.
22. $3x-2y-4$ and $-\frac{5}{6}x$.
23. $\frac{2}{3}a-\frac{1}{6}b-c$ and $\frac{3}{8}ax$.
24. $\frac{6}{7}a^2x^2-\frac{3}{2}ax^3$ and $-\frac{7}{3}a^3x$.
25. $-\frac{5}{3}a^2x^2$ and $-\frac{3}{2}a^2+ax-\frac{3}{5}x^2$.
26. $-\frac{7}{2}xy$ and $-3x^2+\frac{2}{7}xy$.
27. $-\frac{3}{2}x^3y^2$ and $-\frac{1}{3}x^2+2y^2$.
28. $-\frac{4}{7}x^5y^3$ and $\frac{7}{4}x^3-\frac{4}{7}y^3$.

39. The results of Art. 33 may be extended to the case where both of the expressions to be multiplied together contain two or more terms. For instance

$$(a-b+c)m = am - bm + cm;$$

replacing m by $x-y$, we have

$$\begin{aligned}(a-b+c)(x-y) &= a(x-y)-b(x-y)+c(x-y)\\ &= (ax-ay)-(bx-by)+(cx-cy)\\ &= ax-ay-bx+by+cx-cy.\end{aligned}$$

We may now state the general rule for multiplying together any two compound expressions.

Rule. *Multiply each term of the first expression by each term of the second. When the terms multiplied together have like signs, prefix to the product the sign* +, *when unlike prefix* −; *the algebraical sum of the partial products so formed gives the complete product.* This process is called **Distributing the Product.**

40. It should be noticed that the product of $a+b$ and $x-y$ is briefly expressed by $(a+b)(x-y)$, in which the brackets indicate that the expression $a+b$ taken as a whole is to be multiplied by the expression $x-y$ taken as a whole. By the above rule, the value of the product is the algebraical sum of the partial products $+ax$, $+bx$, $-ay$, $-by$; the sign of each product being determined by the rule of signs.

Example 1. Multiply $x+8$ by $x+7$.

The product
$$\begin{aligned} &= (x+8)(x+7) \\ &= x^2+8x+7x+56 \\ &= x^2+15x+56. \end{aligned}$$

The operation is more conveniently arranged as follows:

$$\begin{array}{rl} & x+8 \\ & \underline{x+7} \\ & x^2+8x \\ & \underline{+7x+56} \\ \text{by addition,} & x^2+15x+56 \end{array}$$

We begin on the left and work to the right, placing the second result one place to the right, so that like terms may stand in the same vertical column.

Example 2. Multiply $2x-3y$ by $4x-7y$.

$$\begin{array}{rl} & 2x-3y \\ & \underline{4x-7y} \\ & 8x^2-12xy \\ & \underline{-14xy+21y^2} \\ \text{by addition,} & 8x^2-26xy+21y^2. \end{array}$$

EXAMPLES V. d.

Find the product of

1. $x+5$ and $x+10$.
2. $x+5$ and $x-5$.
3. $x-7$ and $x-10$.
4. $x-7$ and $x+10$.
5. $x+7$ and $x-10$.
6. $x+7$ and $x+10$.
7. $x+6$ and $x-6$.
8. $x+8$ and $x-4$.
9. $x-12$ and $x-1$.
10. $x+12$ and $x-1$.
11. $x-15$ and $x+15$.
12. $x-15$ and $-x+3$.
13. $-x-2$ and $-x-3$.
14. $-x+7$ and $x-7$.
15. $-x+5$ and $-x-5$.
16. $x-13$ and $x+14$.
17. $x-17$ and $x+18$.
18. $x+19$ and $x-20$.
19. $-x-16$ and $-x+16$.
20. $-x+21$ and $x-21$.
21. $2x-3$ and $x+8$.
22. $2x+3$ and $x-8$.

Find the product of

23.	$x-5$ and $2x-1$.	**24.**	$2x-5$ and $x-1$.
25.	$3x-5$ and $2x+7$.	**26.**	$3x+5$ and $2x-7$.
27.	$5x-6$ and $2x+3$.	**28.**	$5x+6$ and $2x-3$.
29.	$3x-5y$ and $3x+5y$.	**30.**	$3x-5y$ and $3x-5y$.
31.	$a-2b$ and $a+3b$.	**32.**	$a-7b$ and $a+8b$.
33.	$3a-6b$ and $a-8b$.	**34.**	$a-9b$ and $a+5b$.
35.	$x+a$ and $x-b$.	**36.**	$x-a$ and $x+b$.
37.	$x-2a$ and $x+3b$.	**38.**	$ax-by$ and $ax+by$.
39.	$xy-ab$ and $xy+ab$.	**40.**	$2pq-3r$ and $2pq+3r$.

[*With the exception of* Art. 44, *the rest of this chapter may be postponed and taken after* Chapter XIV.]

***41.** We shall now give a few examples of greater difficulty.

Example 1. Find the product of $3x^2-2x-5$ and $2x-5$.

$$\begin{array}{rrrr}
3x^2 & -\ 2x & -5 & \\
2x & -\ 5 & & \\
\hline
6x^3 & -\ 4x^2 & -10x & \\
 & -15x^2 & +10x & +25 \\
\hline
6x^3 & -19x^2 & & +25
\end{array}$$

Each term of the first expression is multiplied by $2x$, the first term of the second expression; then each term of the first expression is multiplied by -5; like terms are placed in the same columns and the results added.

Example 2. Multiply $a-b+3c$ by $a+2b$.

$$\begin{array}{lllll}
a^{} & -\ b & +3c & & \\
a & +\ 2b & & & \\
\hline
a^2 & -\ ab & +3ac & & \\
 & \ 2ab & & -2b^2 & +6bc \\
\hline
a^2 & +\ ab & +3ac & -2b^2 & +6bc
\end{array}$$

***42.** When the coefficients are fractional we use the ordinary process of Multiplication, combining the fractional coefficients by the rules of Arithmetic.

Example. Multiply $\frac{1}{3}a^2-\frac{1}{2}ab+\frac{2}{3}b^2$ by $\frac{1}{2}a+\frac{1}{3}b$.

$$\begin{array}{lllll}
\frac{1}{3}a^2 & -\ \frac{1}{2}ab & +\frac{2}{3}b^2 & & \\
\frac{1}{2}a & +\ \frac{1}{3}b & & & \\
\hline
\frac{1}{6}a^3 & -\ \frac{1}{4}a^2b & +\frac{1}{3}ab^2 & & \\
 & +\ \frac{1}{9}a^2b & -\frac{1}{6}ab^2 & +\frac{2}{9}b^3 & \\
\hline
\frac{1}{6}a^3 & -\frac{5}{36}a^2b & +\frac{1}{6}ab^2 & +\frac{2}{9}b^3 &
\end{array}$$

*** 43.** If the expressions are not arranged according to powers, ascending or descending, of some common letter, a rearrangement will be found convenient.

Example 1. Find the product of $2a^2+4b^2-3ab$ and $3ab-5a^2+4b^2$

$$\begin{array}{rrrrr} 2a^2 & -3ab & +4b^2 & & \\ -5a^2 & +3ab & +4b^2 & & \\ \hline -10a^4 & +15a^3b & -20a^2b^2 & & \\ & +6a^3b & -9a^2b^2 & +12ab^3 & \\ & & 8a^2b^2 & -12ab^3 & +16b^4 \\ \hline -10a^4 & +21a^3b & -21a^2b^2 & & +16b^4 \end{array}$$

The rearrangement is not *necessary*, but convenient, because it makes the collection of like terms more easy.

Example 2. Multiply $2xz-z^2+2x^2-3yz+xy$ by $x-y+2z$.

$$\begin{array}{rrrrrrrrr} 2x^2 & +xy & +2xz & -3yz & -z^2 & & & & \\ x & -y & +2z & & & & & & \\ \hline 2x^3 & +x^2y & +2x^2z & -3xyz & -xz^2 & & & & \\ & -2x^2y & & -2xyz & & -xy^2 & +3y^2z & +yz^2 & \\ & & 4x^2z & +2xyz & +4xz^2 & & & -6yz^2 & -2z^3 \\ \hline 2x^3 & -x^2y & +6x^2z & -3xyz & +3xz^2 & -xy^2 & +3y^2z & -5yz^2 & -2z^3 \end{array}$$

* EXAMPLES V. e.

Multiply together

1. $a+b+c,\ a+b-c.$
2. $a-2b+c,\ a+2b-c.$
3. $a^2-ab+b^2,\ a^2+ab+b^2.$
4. $x^2+3y^2,\ x+4y.$
5. $x^3-2x^2+8,\ x+2.$
6. $x^4-x^2y^2+y^4,\ x^2+y^2.$
7. $x^2+xy+y^2,\ x-y.$
8. $a^2-2ax+4x^2,\ a^2+2ax+4x^2.$
9. $16a^2+12ab+9b^2,\ 4a-3b.$
10. $a^2x-ax^2+x^3-a^3,\ x+a.$
11. $x^2+x-2,\ x^2+x-6.$
12. $2x^3-3x^2+2x,\ 2x^2+3x+2.$
13. $-a^5+a^4b-a^3b^2,\ -a-b.$
14. $x^3-7x+5,\ x^2-2x+3.$
15. $a^3+2a^2b+2ab^2,\ a^2-2ab+2b^2.$
16. $4x^2+6xy+9y^2,\ 2x-3y.$
17. $x^2-3xy-y^2,\ -x^2+xy+y^2.$
18. $b^3-a^2b^2+a^3,\ a^3+a^2b^2+b^3.$
19. $x^2-2xy+y^2,\ x^2+2xy+y^2.$
20. $ab+cd+ac+bd,\ ab+cd-ac-bd.$
21. $-3a^2b^2+4ab^3+15a^3b,\ 5a^2b^2+ab^3-3b^4.$
22. $27x^3-36ax^2+48a^2x-64a^3,\ 3x+4a.$
23. $a^2-5ab-b^2,\ a^2+5ab+b^2.$
24. $x^2-xy+x+y^2+y+1,\ x+y-1.$

Multiply together

25. $a^2+b^2+c^2-bc-ca-ab,\ a+b+c.$

26. $-x^3y+y^4+x^2y^2+x^4-xy^3,\ x+y.$

27. $x^{12}-x^9y^2+x^6y^4-x^3y^6+y^8,\ x^3+y^2.$

28. $3a^2+2a+2a^3+1+a^4,\ a^2-2a+1.$

29. $-ax^2+3axy^2-9ay^4,\ -ax-3ay^2.$

30. $-2x^3y+y^4+3x^2y^2+x^4-2xy^3,\ x^2+2xy+y^2.$

31. $\frac{1}{2}a^2+\frac{1}{3}a+\frac{1}{4},\ \frac{1}{2}a-\frac{1}{3}.$

32. $\frac{1}{2}x^2-2x+\frac{3}{2},\ \frac{1}{2}x+\frac{1}{3}.$

33. $\frac{2}{3}x^2+xy+\frac{3}{2}y^2,\ \frac{1}{3}x-\frac{1}{2}y.$

34. $\frac{3}{2}x^2-ax-\frac{2}{3}a^2,\ \frac{3}{4}x^2-\frac{1}{2}ax+\frac{1}{3}a^2.$

35. $\frac{1}{2}x^2-\frac{2}{3}x-\frac{3}{4},\ \frac{1}{2}x^2+\frac{2}{3}x-\frac{3}{4}.$

36. $\frac{2}{3}ax+\frac{2}{3}x^2+\frac{1}{3}a^2,\ \frac{3}{4}a^2+\frac{3}{2}x^2-\frac{3}{2}ax.$

44. Products written down by inspection. Although the result of multiplying together two binomial factors, such as $x+8$ and $x-7$, can always be obtained by the methods already explained, it is of the utmost importance that the student should soon learn to write down the product rapidly *by inspection.*

This is done by observing in what way the coefficients of the terms in the product arise, and noticing that they result from the combination of the numerical coefficients in the two binomials which are multiplied together; thus

$$\begin{aligned}(x+8)(x+7)&=x^2+8x+7x+56\\&=x^2+15x+56.\\(x-8)(x-7)&=x^2-8x-7x+56\\&=x^2-15x+56.\\(x+8)(x-7)&=x^2+8x-7x-56\\&=x^2+x-56.\\(x-8)(x+7)&=x^2-8x+7x-56\\&=x^2-x-56.\end{aligned}$$

In each of these results we notice that:

1. The product consists of three terms.

2. The first term is the product of the first terms of the two binomial expressions.

3. The third term is the product of the second terms of the two binomial expressions.

4. The middle term has for its coefficient the sum of the numerical quantities (taken with their proper signs) in the second terms of the two binomial expressions.

The intermediate step in the work may be omitted, and the products written down at once, as in the following examples:

$$(x+2)(x+3)=x^2+5x+6.$$
$$(x-3)(x+4)=x^2+x-12.$$
$$(x+6)(x-9)=x^2-3x-54.$$
$$(x-4y)(x-10y)=x^2-14xy+40y^2.$$
$$(x-6y)(x+4y)=x^2-2xy-24y^2.$$

By an easy extension of these principles we may write down the product of *any* two binomials.

Thus
$$\begin{aligned}(2x+3y)(x-y)&=2x^2+3xy-2xy-3y^2\\&=2x^2+xy-3y^2.\\(3x-4y)(2x+y)&=6x^2-8xy+3xy-4y^2\\&=6x^2-5xy-4y^2.\\(x+4)(x-4)&=x^2+4x-4x-16\\&=x^2-16.\\(2x+5y)(2x-5y)&=4x^2+10xy-10xy-25y^2\\&=4x^2-25y^2.\end{aligned}$$

EXAMPLES V. f.

Write down the values of the following products:

1. $(x+8)(x-5)$.
2. $(x+6)(x-1)$.
3. $(x-3)(x+10)$.
4. $(x-1)(x+5)$.
5. $(x+7)(x-9)$.
6. $(x-10)(x-8)$.
7. $(x-4)(x+11)$.
8. $(x-2)(x+4)$.
9. $(x+2)(x-2)$.
10. $(a-1)(a+1)$.
11. $(a+9)(a-5)$.
12. $(a-3)(a+12)$.
13. $(a-8)(a+4)$.
14. $(a-8)(a+8)$.
15. $(a-6)(a+13)$.
16. $(a+3)(a+3)$.
17. $(a-11)(a+11)$.
18. $(a-8)(a-8)$.
19. $(x-3a)(x+2a)$.
20. $(x+6a)(x-5a)$.
21. $(x+3a)(x-3a)$.
22. $(x+4y)(x-2y)$.
23. $(x+7y)(x-7y)$.
24. $(x-3y)(x-3y)$.
25. $(a+3b)(a+3b)$.
26. $(a-5b)(a+10b)$.
27. $(a-9b)(a-8b)$.
28. $(2x-5)(x+2)$.
29. $(2x-5)(x-2)$.
30. $(2x+3)(x-3)$.
31. $(3x-1)(x+1)$.
32. $(2x+5)(2x-1)$.
33. $(3x+7)(2x-3)$.
34. $(4x-3)(2x+3)$.
35. $(3x+8)(3x-8)$.
36. $(2x-5)(2x-5)$.
37. $(3x-2y)(3x+y)$.
38. $(3x+2y)(3x+2y)$.
39. $(2x+7y)(2x-5y)$.
40. $(5x+3a)(5x-3a)$.
41. $(2x-5a)(x+5a)$.
42. $(2x+a)(2x+a)$.

The Method of Detached Coefficients.

***45.** When two compound expressions contain powers of one letter only, the labour of multiplication may be lessened by using **detached coefficients**, that is, by writing down the coefficients only, multiplying them together in the ordinary way, and then inserting the successive powers of the letter at the end of the operation. In using this method the expressions must be arranged according to ascending or descending powers of the common letter, and zero coefficients must be used to represent terms corresponding to missing powers of that letter.

Example. Multiply $2x^3-4x^2-5$ by $3x^2+4x-2$.

$$\begin{array}{rrrrrr}
2- & 4+ & 0- & 5 & & \\
3+ & 4- & 2 & & & \\
\hline
6- & 12+ & 0- & 15 & & \\
 & 8- & 16+ & 0- & 20 & \\
 & & -4+ & 8- & 0+ & 10 \\
\hline
6- & 4- & 20- & 7- & 20+ & 10
\end{array}$$

Here we insert a zero coefficient to represent the power of x which is absent in the multiplicand. In the product the highest power of x is clearly x^5, and the others follow in descending order.

Thus the product is

$$6x^5-4x^4-20x^3-7x^2-20x+10.$$

The method of detached coefficients may also be used to multiply two compound expressions which are homogeneous and contain powers of two letters.

Example. Multiply $3a^4+2a^3b+4ab^3+2b^4$ by $2a^2-b^2$.

$$\begin{array}{l}
3+2+0+4+2 \\
2+0-1 \\
\hline
6+4+0+8+4 \\
\quad\;\; -3-2-0-4-2 \\
\hline
6+4-3+6+4-4-2
\end{array}$$

We write a zero coefficient to represent the term containing a^2b^2 which is absent in the first expression. Similarly, the term containing ab is represented by a zero coefficient in the second expression.

It is easily seen how the powers of a and b arise in the successive terms, and the complete product is

$$6a^6+4a^5b-3a^4b^2+6a^3b^3+4a^2b^4-4ab^5-2b^6.$$

Note. Beginners should on no account attempt to use detached coefficients until they are well practised in the ordinary full process of multiplication.

CHAPTER VI.

DIVISION.

[*If preferred, the articles in this chapter marked with an asterisk may be postponed and taken after* Chapter XV.]

46. WHEN a quantity a is divided by the quantity b, the **quotient** is defined to be that which when multiplied by b produces a. This operation of division is denoted by $a \div b$, $\frac{a}{b}$ or a/b; in each of these modes of expression a is called the **dividend**, and b the **divisor.**

Division is thus the inverse of multiplication, and

$$(a \div b) \times b = a.$$

This statement may also be expressed verbally as follows:

$$\text{quotient} \times \text{divisor} = \text{dividend}.$$

Since Division is the inverse of Multiplication, it follows that the Laws of Commutation, Association, and Distribution, which have been established for Multiplication, hold for Division.

47. *The* **Rule of Signs** *holds for division.*

Thus

$$ab \div a = \frac{ab}{a} = \frac{a \times b}{a} = b.$$

$$-ab \div a = \frac{-ab}{a} = \frac{a \times (-b)}{a} = -b.$$

$$ab \div (-a) = \frac{ab}{-a} = \frac{(-a) \times (-b)}{-a} = -b.$$

$$-ab \div (-a) = \frac{-ab}{-a} = \frac{(-a) \times b}{-a} = b.$$

Hence in division as well as multiplication

like signs produce $+$,

unlike signs produce $-$.

Division of Simple Expressions.

48. The method is shewn in the following examples:

Example 1. Since the product of 4 and x is $4x$, it follows that when $4x$ is divided by x the quotient is 4,

or otherwise, $$4x \div x = 4.$$

Example 2. Divide $27a^5$ by $9a^3$.

$$\text{The quotient} = \frac{27a^5}{9a^3} = \frac{27aaaaa}{9aaa}$$
$$= 3aa = 3a^2.$$

We remove from the divisor and dividend the factors common to both, just as in arithmetic.

Therefore $$27a^5 \div 9a^3 = 3a^2.$$

Example 3. Divide $35a^3b^2c^3$ by $7ab^2c^2$.

$$\text{The quotient} = \frac{35aaa \, . \, bb \, . \, ccc}{7a \, . \, bb \, . \, cc} = 5aa \, . \, c = 5a^2c.$$

We see, in each case, that *the index of any letter in the quotient is the difference of the indices of that letter in the dividend and divisor.* This is called the **Index Law for Division.**

The rule may now be stated:

Rule. *The index of each letter in the quotient is obtained by subtracting the index of that letter in the divisor from that in the dividend.*

To the result so obtained prefix with its proper sign the quotient of the coefficient of the dividend by that of the divisor.

Example 4. Divide $45a^6b^2x^4$ by $-9a^3bx^2$.

$$\text{The quotient} = (-5) \times a^{6-3}b^{2-1}x^{4-2}$$
$$= -5a^3bx^2.$$

Example 5. $-21a^2b^3 \div (-7a^2b^2) = 3b$.

Note. If we apply the rule to divide any power of a letter by the same power of the letter we are led to a curious conclusion.

Thus, by the rule $$a^3 \div a^3 = a^{3-3} = a^0;$$

but also $$a^3 \div a^3 = \frac{a^3}{a^3} = 1.$$

$$\therefore \quad a^0 = 1.$$

This result will appear somewhat strange to the beginner, but its full significance will be explained in the chapter on the Theory of Indices.

Division of a Compound Expression by a Simple Expression.

49. Rule. *To divide a compound expression by a single factor, divide each term separately by that factor, and take the algebraic sum of the partial quotients so obtained.*

This follows at once from Art. 33.

Examples. (1) $(9x-12y+3z)\div -3 = -3x+4y-z.$

(2) $(36a^3b^2-24a^2b^5-20a^4b^2)\div 4a^2b = 9ab-6b^4-5a^2b.$

(3) $(2x^2-5xy+\frac{3}{2}x^2y^3)\div -\frac{1}{2}x = -4x+10y-3xy^3.$

EXAMPLES VI. a.

Divide

1. $3x^3$ by x^2.

2. $27x^4$ by $-9x^3$.

3. $-35x^6$ by $7x^3$.

4. abx^2 by $-ax$.

5. x^3y^3 by x^2y.

6. a^4x^3 by $-a^2x^3$.

7. $4a^2b^2c^3$ by ab^2c^2.

8. $12a^6b^6c^6$ by $-3a^4b^2c$.

9. $-a^5c^9$ by $-ac^3$.

10. $15x^5y^7z^4$ by $5x^2y^2z^2$.

11. $-16x^3y^2$ by $-4xy^2$.

12. $-48a^9$ by $-8a^3$.

13. $35a^{11}$ by $7a^7$.

14. $63a^7b^8c^3$ by $9a^5b^5c^3$.

15. $7a^2bc$ by $-7a^2bc$.

16. $28a^4b^3$ by $-4a^3b$.

17. $16b^2yx^2$ by $-2xy$.

18. $-50y^3x^3$ by $-5x^3y$.

19. x^2-2xy by x.

20. x^3-3x^2+x by x.

21. $x^6-7x^5+4x^4$ by x^2.

22. $10x^7-8x^6+3x^4$ by x^3.

23. $15x^5-25x^4$ by $-5x^3$.

24. $27x^6-36x^5$ by $9x^5$.

25. $-24x^6-32x^4$ by $-8x^3$.

26. $34x^3y^2-51x^2y^3$ by $17xy$.

27. $a^2-ab-ac$ by $-a$.

28. $a^3-a^2b-a^2b^2$ by a^2.

29. $3x^3-9x^2y-12xy^2$ by $-3x$.

30. $4x^4y^4-8x^3y^2+6xy^3$ by $-2xy$.

31. $-3a^2+\frac{9}{2}ab-6ac$ by $-\frac{3}{2}a$.

32. $\frac{1}{2}x^5y^2-3x^3y^4$ by $-\frac{3}{2}x^3y^2$.

33. $-\frac{5}{2}x^2+\frac{5}{3}xy+\frac{10}{3}x$ by $-\frac{5}{6}x$.

34. $-2a^5x^3+\frac{7}{2}a^4x^4$ by $\frac{7}{3}a^3x$.

35. $\frac{1}{4}a^2x-\frac{1}{16}abx-\frac{3}{8}acx$ by $\frac{3}{8}ax$.

Division of Compound Expressions.

50. To divide one compound expression by another.

Rule. 1. *Arrange divisor and dividend in ascending or descending powers of some common letter.*

2. *Divide the term on the left of the dividend by the term on the left of the divisor, and put the result in the quotient.*

3. *Multiply the* WHOLE *divisor by this quotient, and put the product under the dividend.*

4. *Subtract and bring down from the dividend as many terms as may be necessary.*

Repeat these operations till all the terms from the dividend are brought down.

Example 1. Divide $x^2+11x+30$ by $x+6$.

Arrange the work thus:

$$x+6\,)\,x^2+11x+30\,($$

divide x^2, the first term of the dividend, by x, the first term of the divisor; the quotient is x. Multiply the *whole* divisor by x, and put the product x^2+6x under the dividend. We then have

$$\begin{array}{rl} x+6\,) & x^2+11x+30\,(\,x \\ & \underline{x^2+\ \ 6x} \\ \text{by subtraction} & \qquad 5x+30. \end{array}$$

On repeating the process above explained we find that the next term in the quotient is $+5$.

The entire operation is more compactly written as follows:

$$\begin{array}{rl} x+6\,) & x^2+11x+30\,(\,x+5 \\ & \underline{x^2+\ \ 6x} \\ & \qquad 5x+30 \\ & \qquad \underline{5x+30} \end{array}$$

The reason for the rule is this: the dividend may be divided into as many parts as may be convenient, and the complete quotient is found by taking the sum of all the partial quotients. Thus $x^2+11x+30$ is divided by the above process into two parts, namely x^2+6x, and $5x+30$, and each of these is divided by $x+6$; thus we obtain the complete quotient $x+5$.

Example 2. Divide $24x^2-65xy+21y^2$ by $8x-3y$.

$$\begin{array}{rl} 8x-3y\,) & 24x^2-65xy+21y^2\,(\,3x-7y \\ & \underline{24x^2-\ \ 9xy} \\ & \qquad -56xy+21y^2 \\ & \qquad \underline{-56xy+21y^2} \end{array}$$

EXAMPLES VI. b.

Divide

1. x^2+3x+2 by $x+1$.

2. $x^2-7x+12$ by $x-3$.

3. $a^2-11a+30$ by $a-5$.

4. $a^2-49a+600$ by $a-25$.

5. $3x^2+10x+3$ by $x+3$.

6. $2x^2+11x+5$ by $2x+1$.

7. $5x^2+11x+2$ by $x+2$.

8. $2x^2+17x+21$ by $2x+3$.

9. $5x^2+16x+3$ by $x+3$.

10. $3x^2+34x+11$ by $3x+1$.

11. $4x^2+23x+15$ by $4x+3$.

12. $6x^2-7x-3$ by $2x-3$.

13. $3x^2+x-14$ by $x-2$.

14. $3x^2-x-14$ by $x+2$.

15. $6x^2-31x+35$ by $2x-7$.

16. $4x^2+x-14$ by $x+2$.

17. $12a^2-7ax-12x^2$ by $3a-4x$.

18. $15a^2+17ax-4x^2$ by $3a+4x$.

19. $12a^2-11ac-36c^2$ by $4a-9c$.

20. $9a^2+6ac-35c^2$ by $3a+7c$.

21. $60x^2-4xy-45y^2$ by $10x-9y$.

22. $96x^2-15y^2-4xy$ by $12x-5y$.

23. $7x^3+96x^2-28x$ by $7x-2$.

24. $100x^3-3x-13x^2$ by $3+25x$.

25. $27x^3+9x^2-3x-10$ by $3x-2$.

26. $16a^3-46a^2+39a-9$ by $8a-3$.

27. $15+3a-7a^2-4a^3$ by $5-4a$.

28. $16-96x+216x^2-216x^3+81x^4$ by $2-3x$.

*51. The process of Art. 50 is applicable to cases in which the divisor consists of more than two terms.

Example 1. Divide $6x^5-x^4+4x^3-5x^2-x-15$ by $2x^2-x+3$.

$$
\begin{array}{r|l}
2x^2-x+3 \,) & 6x^5-x^4+4x^3-5x^2-x-15\ (\,3x^3+x^2-2x-5 \\
& \underline{6x^5-3x^4+9x^3} \\
& \qquad 2x^4-5x^3-5x^2 \\
& \qquad \underline{2x^4-x^3+3x^2} \\
& \qquad\quad -4x^3-8x^2-x \\
& \qquad\quad \underline{-4x^3+2x^2-6x} \\
& \qquad\qquad -10x^2+5x-15 \\
& \qquad\qquad \underline{-10x^2+5x-15}
\end{array}
$$

Example 2. Divide $2a^3+10-16a-39a^2+15a^4$ by $2-4a-5a^2$.

Arrange the expressions in *ascending* powers of a and use detached coefficients as in Art. 45.

$$
\begin{array}{r|l}
2-4-5\,) & 10-16-39+2+15\ (\,5+2-3 \\
& \underline{10-20-25} \\
& \qquad 4-14+2 \\
& \qquad \underline{4-8-10} \\
& \qquad\quad -6+12+15 \\
& \qquad\quad \underline{-6+12+15}
\end{array}
$$

Thus the quotient is $5+2a-3a^2$.

***52.** We add a few harder cases worked out in full.

Example 1. Divide x^4+4a^4 by $x^2+2xa+2a^2$.

$$\begin{array}{l}
x^2+2xa+2a^2\,)\,x^4+4a^4\,(\,x^2-2xa+2a^2\\
\qquad\qquad\qquad\quad \underline{x^4+2x^3a+2x^2a^2}\\
\qquad\qquad\qquad\qquad -2x^3a-2x^2a^2\\
\qquad\qquad\qquad\qquad \underline{-2x^3a-4x^2a^2-4xa^3}\\
\qquad\qquad\qquad\qquad\qquad\qquad 2x^2a^2+4xa^3+4a^4\\
\qquad\qquad\qquad\qquad\qquad\qquad \underline{2x^2a^2+4xa^3+4a^4}
\end{array}$$

Example 2. Divide $a^3+b^3+c^3-3abc$ by $a+b+c$.

$$\begin{array}{l}
a+b+c\,)\,a^3-3abc+b^3+c^3\,(\,a^2-ab-ac+b^2-bc+c^2\\
\qquad\qquad \underline{a^3+a^2b+a^2c}\\
\qquad\qquad -a^2b-a^2c-3abc\\
\qquad\qquad \underline{-a^2b-ab^2-abc}\\
\qquad\qquad\qquad -a^2c+ab^2-2abc\\
\qquad\qquad\qquad \underline{-a^2c\qquad\quad -abc-ac^2}\\
\qquad\qquad\qquad\qquad ab^2-abc+ac^2+b^3\\
\qquad\qquad\qquad\qquad \underline{ab^2\qquad\qquad\quad +b^3+b^2c}\\
\qquad\qquad\qquad\qquad\quad -abc+ac^2-b^2c\\
\qquad\qquad\qquad\qquad\quad \underline{-abc\qquad\;\, -b^2c-bc^2}\\
\qquad\qquad\qquad\qquad\qquad\qquad ac^2+bc^2+c^3\\
\qquad\qquad\qquad\qquad\qquad\qquad \underline{ac^2+bc^2+c^3}
\end{array}$$

Note. In the above example the dividend and successive remainders are arranged in *descending* powers of a.

The result of this important division will be referred to later.

***53.** When the coefficients are fractional the ordinary process may still be employed.

Example. Divide $\frac{1}{4}x^3+\frac{1}{72}xy^2+\frac{1}{12}y^3$ by $\frac{1}{2}x+\frac{1}{3}y$.

$$\begin{array}{l}
\frac{1}{2}x+\frac{1}{3}y\,)\,\frac{1}{4}x^3+\frac{1}{72}xy^2+\frac{1}{12}y^3\,(\,\frac{1}{2}x^2-\frac{1}{3}xy+\frac{1}{4}y^2\\
\qquad\qquad \underline{\frac{1}{4}x^3+\frac{1}{6}x^2y}\\
\qquad\qquad\quad -\frac{1}{6}x^2y+\frac{1}{72}xy^2\\
\qquad\qquad\quad \underline{-\frac{1}{6}x^2y-\frac{1}{9}xy^2}\\
\qquad\qquad\qquad\qquad \frac{1}{8}xy^2+\frac{1}{12}y^3\\
\qquad\qquad\qquad\qquad \underline{\frac{1}{8}xy^2+\frac{1}{12}y^3}
\end{array}$$

***54.** In the examples given hitherto the divisor has been exactly contained in the dividend. When the division is not exact the work should be carried on until the remainder is of lower dimensions [Art. 24] than the divisor.

*EXAMPLES VI. c.

[*Examples* 1–20 *will furnish practice in the use of Detached Coefficients as explained in* Art. 51.]

Divide

1. $x^3-x^2-9x-12$ by x^2+3x+3.

2. $2y^3-3y^2-6y-1$ by $2y^2-5y-1$.

3. $6m^3-m^2-14m+3$ by $3m^2+4m-1$.

4. $6a^5-13a^4+4a^3+3a^2$ by $3a^3-2a^2-a$.

5. $x^4+x^3+7x^2-6x+8$ by x^2+2x+8.

6. $a^4-a^3-8a^2+12a-9$ by a^2+2a-3.

7. $a^4+6a^3+13a^2+12a+4$ by a^2+3a+2.

8. $2x^4-x^3+4x^2+7x+1$ by x^2-x+3.

9. $x^5-5x^4+9x^3-6x^2-x+2$ by x^2-3x+2.

10. $x^5-4x^4+3x^3+3x^2-3x+2$ by x^2-x-2.

11. $30x^4+11x^3-82x^2-5x+3$ by $2x-4+3x^2$.

12. $30y+9-71y^3+28y^4-35y^2$ by $4y^2-13y+6$.

13. $6k^5-15k^4+4k^3+7k^2-7k+2$ by $3k^3-k+1$.

14. $15+2m^4-31m+9m^2+4m^3+m^5$ by $3-2m-m^2$.

15. $2x^3-8x+x^4+12-7x^2$ by x^2+2-3x.

16. $x^5-2x^4-4x^3+19x^2$ by x^3-7x+5.

17. $192-x^4+128x+4x^2-8x^3$ by $16-x^2$.

18. $14x^4+45x^3y+78x^2y^2+45xy^3+14y^4$ by $2x^2+5xy+7y^2$.

19. $x^5-x^4y+x^3y^2-x^3+x^2-y^3$ by x^3-x-y.

20. $x^5+x^4y-x^3y^2+x^3-2xy^2+y^3$ by x^2+xy-y^2.

21. a^9-b^9 by a^3-b^3.

22. x^9-y^9 by x^2+xy+y^2.

23. $x^7-2y^{14}-7x^5y^4-7xy^{12}+14x^3y^8$ by $x-2y^2$.

24. $a^3+3a^2b+b^3-1+3ab^2$ by $a+b-1$.

25. x^8-y^8 by $x^3+x^2y+xy^2+y^3$.

26. $a^{12}-b^{12}$ by a^2-b^2.

27. $a^{12}+2a^6b^6+b^{12}$ by $a^4+2a^2b^2+b^4$.

28. $1-a^3-8x^3-6ax$ by $1-a-2x$.

Find the quotient of

29. $\frac{1}{8}a^3-\frac{9}{4}a^2x+\frac{27}{2}ax^2-27x^3$ by $\frac{1}{2}a-3x$.

30. $\frac{1}{27}a^3-\frac{1}{12}a^2+\frac{1}{16}a-\frac{1}{64}$ by $\frac{1}{3}a-\frac{1}{4}$.

31. $\frac{3}{4}a^2c^3+\frac{6}{125}a^5$ by $\frac{1}{5}a^2+\frac{1}{2}ac$.

32. $\frac{9}{16}a^4-\frac{3}{4}a^3-\frac{7}{4}a^2+\frac{4}{3}a+\frac{16}{9}$ by $\frac{3}{2}a^2-\frac{8}{3}-a$.

33. $36x^2+\frac{1}{9}y^2+\frac{1}{4}-4xy-6x+\frac{1}{3}y$ by $6x-\frac{1}{3}y-\frac{1}{2}$.

34. $\frac{8}{27}a^5-\frac{243}{512}ax^4$ by $\frac{2}{3}a-\frac{3}{4}x$.

*** 55.** The following examples in division may be easily verified; they are of great importance, and should be carefully noticed.

$$\text{I.}\quad\begin{cases}\dfrac{x^2-y^2}{x-y}=x+y,\\[2mm] \dfrac{x^3-y^3}{x-y}=x^2+xy+y^2,\\[2mm] \dfrac{x^4-y^4}{x-y}=x^3+x^2y+xy^2+y^3,\end{cases}$$

and so on; the divisor being $x-y$, the terms in the quotient *all positive*, and the index in the dividend *either odd or even.*

$$\text{II.}\quad\begin{cases}\dfrac{x^3+y^3}{x+y}=x^2-xy+y^2,\\[2mm] \dfrac{x^5+y^5}{x+y}=x^4-x^3y+x^2y^2-xy^3+y^4,\\[2mm] \dfrac{x^7+y^7}{x+y}=x^6-x^5y+x^4y^2-x^3y^3+x^2y^4-xy^5+y^6,\end{cases}$$

and so on; the divisor being $x+y$, the terms in the quotient *alternately positive and negative*, and the index in the dividend *always odd.*

$$\text{III.}\quad\begin{cases}\dfrac{x^2-y^2}{x+y}=x-y,\\[2mm] \dfrac{x^4-y^4}{x+y}=x^3-x^2y+xy^2-y^3,\\[2mm] \dfrac{x^6-y^6}{x+y}=x^5-x^4y+x^3y^2-x^2y^3+xy^4-y^5,\end{cases}$$

and so on; the divisor being $x+y$, the terms in the quotient *alternately positive and negative*, and the index in the dividend *always even.*

IV. The expressions x^2+y^2, x^4+y^4, x^6+y^6, ... (where the index is *even*, and the terms *both positive*) are *never* divisible by $x+y$ or $x-y$.

All these different cases may be more concisely stated as follows:

(1) x^n-y^n is divisible by $x-y$ if n be *any* whole number.
(2) x^n+y^n is divisible by $x+y$ if n be any *odd* whole number.
(3) x^n-y^n is divisible by $x+y$ if n be any *even* whole number.
(4) x^n+y^n is never divisible by $x+y$ or $x-y$, when n is an *even* whole number.

CHAPTER VII.

Removal and Insertion of Brackets.

56. Quantities are enclosed within brackets to indicate that they must all be operated upon in the same way. Thus in the expression $2a-3b-(4a-2b)$ the brackets indicate that the expression $4a-2b$ *treated as a whole* has to be subtracted from $2a-3b$. When we wish to enclose within brackets part of an expression already enclosed within brackets it is usual to employ brackets of different forms. The brackets in common use are ( ), { }, [ ]. Sometimes a line called a **vinculum** is drawn over the symbols to be connected; thus $a-\overline{b+c}$ is used with the same meaning as $a-(b+c)$, and hence $a-\overline{b+c}=a-b-c$.

Removal of Brackets.

57. To remove brackets it is usually best to begin with the inside pair, and in dealing with each pair in succession we apply the rules already given in Arts. 21, 22.

Example 1. Simplify, by removing brackets, the expression

$$a-2b-[4a-6b-\{3a-c+(5a-2b-\overline{3a-c+2b})\}].$$

Removing the brackets one by one, we have

$$\begin{aligned}
&a-2b-[4a-6b-\{3a-c+(5a-2b-3a+c-2b)\}]\\
&=a-2b-[4a-6b-\{3a-c+\ 5a-2b-3a+c-2b\}]\\
&=a-2b-[4a-6b-\ 3a+c-\ 5a+2b+3a-c+2b]\\
&=a-2b-\ 4a+6b+\ 3a-c+\ 5a-2b-3a+c-2b\\
&=2a, \text{ by collecting like terms.}
\end{aligned}$$

Example 2. Simplify the expression

$$-[-2x-\{3y-(2x-3y)+(3x-2y)\}+2x].$$

$$\begin{aligned}
\text{The expression}&=-[-2x-\{3y-2x+3y+3x-2y\}+2x]\\
&=-[-2x-\ 3y+2x-3y-3x+2y\ +2x]\\
&=2x+3y-2x+3y+3x-2y-2x\\
&=x+4y.
\end{aligned}$$

EXAMPLES VII. a.

Simplify by removing brackets

1. $a-(b-c)+a+(b-c)+b-(c+a)$.
2. $a-[b+\{a-(b+a)\}]$.
3. $a-[2a-\{3b-(4c-2a)\}]$.
4. $\{a-(b-c)\}+\{b-(c-a)\}-\{c-(a-b)\}$.
5. $2a-(5b+[3c-a])-(5a-[b+c])$.
6. $-\{-[-(a-\overline{b-c})]\}$.
7. $-[a-\{b-(c-a)\}]-[b-\{c-(a-b)\}]$.
8. $-(-(-(-x)))-(-(-y))$.
9. $-[-\{-(b+c-a)\}]+[-\{-(c+a-b)\}]$.
10. $-5x-[3y-\{2x-(2y-x)\}]$.
11. $-(-(-a))-(-(-(-x)))$.
12. $3a-[a+b-\{a+b+c-(a+b+c+d)\}]$.
13. $-2a-[3x+\{3c-(4y+3x+2a)\}]$.
14. $3x-[5y-\{6z-(4x-7y)\}]$.
15. $-[5x-(11y-3x)]-[5y-(3x-6y)]$.
16. $-[15x-\{14y-(15z+12y)-(10x-15z)\}]$.
17. $8x-\{16y-[3x-(12y-x)-8y]+x\}$.
18. $-[x-\{z+(x-z)-(z-x)-z\}-x]$.
19. $-[a+\{a-(a-x)-(a+x)-a\}-a]$.
20. $-[a-\{a+(x-a)-(x-a)-a\}-2a]$.

58. A coefficient placed before any bracket indicates that every term of the expression within the bracket is to be multiplied by that coefficient.

Note. The line between the numerator and denominator of a fraction is a kind of vinculum. Thus $\dfrac{x-5}{3}$ is equivalent to $\frac{1}{3}(x+5)$.

Again, an expression of the form $\sqrt{}(x+y)$ is often written $\sqrt{x+y}$, the line above being regarded as a vinculum indicating the square root of the compound expression $x+y$ *taken as a whole.*

Thus $$\sqrt{25+144}=\sqrt{169}=13,$$

whereas $$\sqrt{25}+\sqrt{144}=5+12=17.$$

59. Sometimes it is advisable to simplify in the course of the work.

Example. Find the value of

$$84-7[-11x-4\{-17x+3(8-\overline{9-5x})\}].$$

$$\begin{aligned}\text{The expression}&=84-7[-11x-4\{-17x+3(8-9+5x)\}]\\&=84-7[-11x-4\{-17x+3(5x-1)\}]\\&=84-7[-11x-4\{-17x+15x-3\}]\\&=84-7[-11x-4\{-2x-3\}]\\&=84-7[-11x+8x+12]\\&=84-7[-3x+12]\\&=84+21x-84\\&=21x.\end{aligned}$$

When the beginner has had a little practice the number of steps may be considerably diminished.

EXAMPLES VII. b.

Simplify by removing brackets

1. $a-[2b+\{3c-3a-(a+b)\}+2a-(b+3c)]$.

2. $a+b-(c+a-[b+c-(a+b-\{c+a-(b+c-a)\})])$.

3. $a-(b-c)-[a-b-c-2\{b+c-3(c-a)-d\}]$.

4. $2x-(3y-4z)-\{2x-(3y+4z)\}-\{3y-(4z+2x)\}$.

5. $b+c-(a+b-[c+a-(b+c-\{a+b-(c+a-b)\})])$.

6. $3b-\{5a-[6a+2(10a-b)]\}$.

7. $a-(b-c)-[a-b-c-2\{b+c\}]$.

8. $3a^2-[6a^2-\{8b^2-(9c^2-2a^2)\}]$.

9. $b-(c-a)-[b-a-c-2\{c+a-3(a-b)-d\}]$.

10. $-20(a-d)+3(b-c)-2[b+c+d-3\{c+d-4(d-a)\}]$.

11. $-4(a+d)+24(b-c)-2[c+d+a-3\{d+a-4(b+c)\}]$.

12. $-10(a+b)-[c+a+b-3\{a+2b-(c+a-b)\}]+4c$.

13. $a-2(b-c)-[-\{-(4a-b-c-2\{a+b+c\})\}]$.

14. $8(b-c)-[-\{a-b-3(c-b+a)\}]$.

15. $2(3b-5a)-7[a-6\{2-5(a-b)\}]$.

16. $6\{a-2[b-3(c+d)]\}-4\{a-3[b-4(c-d)]\}$.

17. $5\{a-2[a-2(a+x)]\}-4\{a-2[a-2(a+x)]\}$.

18. $-10\{a-6[a-(b-c)]\}+60\{b-(c+a)\}$.

19. $-3\{-2[-4(-a)]\}+5\{-2[-2(-a)]\}$.

20. $-2\{-[-(x-y)]\}+\{-2[-(x-y)]\}$.

21. $\frac{1}{4}\{a-5(b-a)\}-\frac{3}{2}\left\{\frac{1}{3}\left(b-\frac{a}{3}\right)-\frac{2}{9}\left[a-\frac{3}{4}\left(b-\frac{4a}{5}\right)\right]\right\}$.

22. $35\left[\frac{3x-4y}{5}-\frac{1}{10}\left\{3x-\frac{5}{7}(7x-4y)\right\}\right]+8(y-2x)$.

23. $\frac{3}{8}\left\{\frac{4}{3}(a-b)-8(b-c)\right\}-\left\{\frac{b-c}{2}-\frac{c-a}{3}\right\}-\frac{1}{2}\left\{c-a-\frac{2}{3}(a-b)\right\}$.

24. $\frac{1}{2}x-\frac{1}{2}\left(\frac{2}{3}y-\frac{1}{2}z\right)-\left[x-\left\{\frac{1}{2}x-\left(\frac{1}{3}y-\frac{1}{4}z\right)\right\}-\left(\frac{2}{3}y-\frac{1}{2}z\right)\right]$.

Insertion of Brackets.

60. The converse operation of inserting brackets is important. The rules for doing this have been enunciated in Arts. 21, 22; for convenience we repeat them.

(1) *Any part of an expression may be enclosed within brackets and the sign + prefixed, the sign of every term within the brackets remaining unaltered.*

(2) *Any part of an expression may be enclosed within brackets and the sign − prefixed, provided the sign of every term within the brackets be changed.*

Examples. $a-b+c-d-e=a-b+(c-d-e)$.

$a-b+c-d-e=a-(b-c)-(d+e)$.

$x^2-ax+bx-ab=(x^2-ax)+(bx-ab)$.

$xy-ax-by+ab=(xy-by)-(ax-ab)$.

61. The terms of an expression can be bracketed in various ways.

Example. The expression $ax-bx+cx-ay+by-cy$

may be written $(ax-bx)+(cx-ay)+(by-cy)$,

or $(ax-bx+cx)-(ay-by+cy)$,

or $(ax-ay)-(bx-by)+(cx-cy)$.

62. Whenever a factor is common to every term within a bracket, it may be removed and placed outside as a multiplier of the expression within the bracket.

Example 1. In the expression

$$ax^3 - cx + 7 - dx^2 + bx - c - dx^3 + bx^2 - 2x$$

bracket together the powers of x so as to have the sign + before each bracket.

$$\begin{aligned}\text{The expression} &= (ax^3 - dx^3) + (bx^2 - dx^2) + (bx - cx - 2x) + (7 - c)\\ &= x^3(a-d) + x^2(b-d) + x(b-c-2) + (7-c)\\ &= (a-d)x^3 + (b-d)x^2 + (b-c-2)x + 7 - c.\end{aligned}$$

In this last result the compound expressions $a-d$, $b-d$, $b-c-2$ are regarded as the coefficients of x^3, x^2, and x respectively.

Example 2. In the expression $-a^2x - 7a + a^2y + 3 - 2x - ab$ bracket together the powers of a so as to have the sign $-$ before each bracket.

$$\begin{aligned}\text{The expression} &= -(a^2x - a^2y) - (7a + ab) - (2x - 3)\\ &= -a^2(x-y) - a(7+b) - (2x-3)\\ &= -(x-y)a^2 - (7+b)a - (2x-3).\end{aligned}$$

EXAMPLES VII. c.

In the following expressions bracket the powers of x so that the signs before all the brackets shall be positive :

1. $ax^4 + bx^2 + 5 + 2bx - 5x^2 + 2x^4 - 3x.$

2. $3bx^2 - 7 - 2x + ab + 5ax^3 + cx - 4x^2 - bx^3.$

3. $2 - 7x^3 + 5ax^2 - 2cx + 9ax^3 + 7x - 3x^2.$

4. $2cx^5 - 3abx + 4dx - 3bx^4 - a^2x^5 + x^4.$

In the following expressions bracket the powers of x so that the signs before all the brackets shall be negative :

5. $ax^2 + 5x^3 - a^2x^4 - 2bx^3 - 3x^2 - bx^4.$

6. $7x^3 - 3c^2x - abx^5 + 5ax + 7x^5 - abcx^3.$

7. $ax^2 + a^2x^3 - bx^2 - 5x^2 - cx^3.$

8. $3b^2x^4 - bx - ax^4 - cx^4 - 5c^2x - 7x^4.$

Simplify the following expressions, and in each result re-group the terms according to powers of x :

9. $ax^3 - 2cx - [bx^2 - \{cx - dx - (bx^3 + 3cx^2)\} - (cx^2 - bx)].$

10. $5ax^3 - 7(bx - cx^2) - \{6bx^2 - (3ax^2 + 2ax) - 4cx^3\}.$

11. $ax^2 - 3\{-ax^3 + 3bx - 4[\frac{1}{6}cx^3 - \frac{2}{3}(ax - bx^2)]\}.$

12. $x^5 - 4bx^4 - \dfrac{1}{6}\left[12ax - 4\left\{3bx^4 - 9\left(\dfrac{cx}{2} - bx^5\right) - \dfrac{3}{2}ax^4\right\}\right].$

13. $x\{x - b - x(a - bx)\} + ax - x\{x - x(ax - b)\}.$

63. In certain cases of addition, multiplication, etc., of expressions which involve literal coefficients, the results may be more conveniently written by grouping the terms according to powers of some common letter.

Example 1. Add together $ax^3 - 2bx^2 + 3$, $bx - cx^3 - x^2$ and

$$x^3 - ax^2 + cx.$$

$$\begin{aligned}\text{The sum} &= ax^3 - 2bx^2 + 3 + bx - cx^3 - x^2 + x^3 - ax^2 + cx \\ &= ax^3 - cx^3 + x^3 - ax^2 - 2bx^2 - x^2 + bx + cx + 3 \\ &= (a - c + 1)x^3 - (a + 2b + 1)x^2 + (b + c)x + 3.\end{aligned}$$

Example 2. Multiply $ax^2 - 2bx + 3c$ by $px - q$.

$$\begin{aligned}\text{The product} &= (ax^2 - 2bx + 3c)(px - q) \\ &= apx^3 - 2bpx^2 + 3cpx - aqx^2 + 2bqx - 3cq \\ &= apx^3 - (2bp + aq)x^2 + (3cp + 2bq)x - 3cq.\end{aligned}$$

EXAMPLES VII. d.

Add together the following expressions, and in each case arrange the result according to powers of x:

1. $ax^3 - 2cx$, $bx^2 - cx^3$, $cx^2 - x$.

2. $x^2 - x - 1$, $ax^2 - bx^3$, $bx + x^3$.

3. $a^2x^3 - 5x$, $2ax^2 - 5ax^3$, $2x^3 - bx^2 - ax$.

4. $ax^2 + bx - c$, $qx - r - px^2$, $x^2 + 2x + 3$.

5. $px^3 - qx$, $qx^2 - px$, $q - x^3$, $px^2 + qx^3$.

Multiply together the following expressions, and in each case arrange the result according to powers of x:

6. $ax^2 + bx + 1$ and $cx + 2$.

7. $cx^2 - 2x + 3$ and $ax - b$.

8. $ax^2 - bx - c$ and $px + q$.

9. $2x^2 - 3x - 1$ and $bx + c$.

10. $ax^2 - 2bx + 3c$ and $x - 1$.

11. $px^2 - 2x - q$ and $ax - 3$.

12. $x^3 + ax^2 - bx - c$ and $x^3 - ax^2 - bx + c$.

13. $ax^3 - x^2 + 3x - b$ and $ax^3 + x^2 + 3x + b$.

14. $x^4 - ax^3 - bx^2 + cx + d$ and $x^4 + ax^3 - bx^2 - cx + d$.

CHAPTER VIII.

SIMPLE EQUATIONS.

64. AN **equation** is a statement that two algebraical expressions are equal.

Thus (i.) $x+3+x+4=2x+7$, (ii.) $4x+2=14$

are equations.

The parts of an equation separated by the sign of equality are called **members** or **sides** of the equation, and are distinguished as the *right side* and the *left side.*

65. If the two expressions are *always* equal, for *any* values we give to the symbols, the equation is called an **identical equation,** or briefly an **identity.** Thus equation (i.) above is an *identity*, as is easily seen by collecting the terms on the left side.

If two expressions are only equal for a particular value or values of the symbols, the equation is called an **equation of condition,** or more usually an *equation*, simply.

Thus the statement $4x+2=14$ will be found to be true only when $x=3$.

This, then, is an equation in the ordinary sense of the term, and the value 3 is said to **satisfy** the equation. The object of the present chapter is to shew how to find the values which satisfy equations of the simpler kinds.

66. The letter whose value it is required to find in any equation is called the **unknown quantity.** The process of finding its value is called **solving the equation.** The value so found is called the **root** or **solution** of the equation.

67. An equation which, when reduced to a simple form, involves no power of the unknown quantity higher than the first is called a **simple equation.** It is usual to denote the unknown quantity by x.

68. The process of solving a simple equation depends only on the following **axioms**:

1. If to equals we add equals the sums are equal.
2. If from equals we take equals the remainders are equal.
3. If equals are multiplied by equals the products are equal.
4. If equals are divided by equals the quotients are equal.

Example 1. To solve the equation $7x=14$.

Dividing both sides by 7, (Axiom 4) we get

$$x=2.$$

Example 2. Solve the equation $\frac{x}{2}=-6$.

Multiplying both sides by 2, (Axiom 3) we get

$$x=-12.$$

Example 3. Solve the equation

$$7x-2x-x=10-23-15.$$

By collecting terms on each side, we get

$$4x=-28.$$

Dividing by 4, (Axiom 4) we get

$$x=-7.$$

EXAMPLES. *(Oral.)*

Find the values which satisfy the following equations :

1. $3x=18$.	**2.** $4x=12$.	**3.** $6x=12$.	**4.** $7x=-7$.
5. $3x=21$.	**6.** $11x=55$.	**7.** $13x=39$.	**8.** $14x=-42$.
9. $7x=-35$.	**10.** $-5x=30$.	**11.** $-2x=-12$.	**12.** $-3x=21$.
13. $3x=0$.	**14.** $-4x=0$.	**15.** $2x=11$.	**16.** $9x=15$.
17. $51x=39$.	**18.** $3x=-7$.	**19.** $28x=35$.	**20.** $34x=-51$.
21. $\frac{x}{3}=7$.	**22.** $\frac{x}{7}=-3$.	**23.** $-\frac{x}{5}=4$.	**24.** $\frac{x}{6}=0$.

25. $8x+5x-3x=17-9+33-11$.

26. $5x-7x+8x=12-5+7+10$.

27. $-3x-12x+5x=29-2+6-13$.

28. $4x-15x-9x+27x=-28+8-60+17$.

69. In the preceding examples the terms have been so arranged that those involving the unknown quantity have been on one side of the equation and the numerical quantities on the other. We can always arrive at this arrangement by the aid of the axioms.

Example. Solve the equation $3x-8=x+12$.

Subtracting x from both sides, we get

$$3x-x-8=12. \qquad \text{[Axiom 2.]}$$

Adding 8 to both sides, we have

$$3x-x=12+8; \qquad \text{[Axiom 1.]}$$

$$\therefore\ 2x=20;$$

dividing by 2, $\quad x=10. \qquad$ [Axiom 4.]

70. Beginners should **verify**, that is, prove the correctness of their solutions by substituting, in both sides, the value obtained for the unknown quantity.

In the last equation $3x - 8 = x + 12$,

if $x = 10$,

the left side $= 3 \times 10 - 8 = 22$,

and the right side $= 10 + 12 = 22$.

Since these two results are equal the solution is correct.

71. In the following examples some preliminary reduction is necessary.

Example 1. Solve $5(x-3) - 7(6-x) = 24 - 3(8-x) - 3$.

Removing brackets, $5x - 15 - 42 + 7x = 24 - 24 + 3x - 3$;

collecting terms, $12x - 57 = 3x - 3$.

Subtracting $3x$ from each side, we get

$9x - 57 = -3$. [Axiom 2.]

Adding 57 to each side, we have

$9x = 54$. [Axiom 1.]

Dividing by 9, $x = 6$. [Axiom 4.]

[*Verification.* When $x = 6$,

the left side $= 5(6-3) - 7(6-6)$
$= 5 \times 3 - 0 = 15$.

The right side $= 24 - 3(8-6) - 3$
$= 24 - 3 \times 2 - 3$
$= 24 - 9 = 15$.

Thus the solution is correct.]

Example 2. Solve $\frac{4x}{5} - \frac{3}{10} = \frac{x}{5} + \frac{x}{4}$.

Here it is convenient to begin by clearing the equation of fractional coefficients. This can be done by multiplying every term on each side of the equation by the least common multiple of the denominators. [Axiom 3.]

Hence, multiplying throughout by 20,

$16x - 6 = 4x + 5x$.

Subtracting $9x$ from each side,

$7x - 6 = 0$.

Adding 6 to each side, $7x = 6$.

Dividing by 7, $x = \frac{6}{7}$.

[*Verification.* When $x=\frac{6}{7}$,

$$\text{the left side}=\frac{4}{5}\times\frac{6}{7}-\frac{3}{10}=\frac{48-21}{70}=\frac{27}{70}.$$

$$\text{The right side}=\frac{1}{5}\times\frac{6}{7}+\frac{1}{4}\times\frac{6}{7}=\frac{24+30}{140}$$

$$=\frac{54}{140}=\frac{27}{70}.$$

Thus the solution is correct.]

72. The preceding examples have been worked out very fully in every detail for the purpose of impressing on beginners the importance of shewing clearly the meaning of every step of their work in solving simple equations. Each step should occupy a separate line, and each successive process should be referred to one of the fundamental axioms; the object in each case being to gradually reduce the equation until it consists of a single term containing x on one side, and a single known term on the other. The required root is then found by dividing each side by the coefficient of x.

Orderly arrangement should be studied throughout, and in particular, the signs of equality in the several lines should be written neatly in column.

In order to furnish the requisite practice in *method* and *arrangement*, we shall now give an exercise containing easy equations which are free from difficulty in the way of reduction, and which involve little actual work.

EXAMPLES VIII. (1).

Find the value of x which satisfies each of the following equations, and in each case verify the solution.

1. $7x-4=17$. **2.** $3x-5=10$. **3.** $2x+15=23$.

4. $5x-9=21$. **5.** $7x=18-2x$. **6.** $3x=25-2x$.

7. $4x-3=2x+1$. **8.** $5x+2=6x-1$. **9.** $3x+2=4x-3$.

10. $4x-3=3x+4$. **11.** $8x-9=33-4x$. **12.** $5x+3=15-x$.

13. $2x+15=27-4x$. **14.** $7x+11=3x+27$.

15. $15-5x=24-8x$. **16.** $9x+21-4x=46$.

17. $5x+7+4x+11+3x=24$. **18.** $0=9-6x-19+10x$.

19. $7-3x=5+4x+11-16x$. **20.** $-3x-5=-7x+1$.

21. $6x+7-19=7x+13-3x-21$.

22. $3x+4+10x-17=14-23x+16-7x$.

Solve and verify the following equations:

23. $\frac{x}{3}=\frac{5}{6}$.
24. $\frac{x}{5}=\frac{4}{3}$.
25. $\frac{2x}{3}=\frac{5}{12}$.
26. $\frac{4x}{5}=\frac{7}{15}$.
27. $\frac{7x}{6}=\frac{4}{9}$.
28. $\frac{3x}{8}=\frac{5}{9}$.
29. $\frac{1}{2}x-\frac{1}{4}x=x-9$.
30. $\frac{x}{3}-\frac{1}{2}=\frac{x}{5}+1\frac{1}{2}$.
31. $\frac{x}{3}-2\frac{1}{2}=\frac{4x}{9}-\frac{2x}{3}$.
32. $\frac{1}{8}x+\frac{1}{6}x-x=\frac{5}{6}-\frac{1}{2}x$.

73. After enough practice to enforce the reasons for the several steps, the solutions may be presented in a shorter form.

When any term is brought over from one side of an equation to the other it is said to be **transposed**.

We shall now shew that any term may be transposed from one side of an equation to the other by simply writing it down on the opposite side *with its sign changed.*

Consider the equation $3x-8=x+12$.

Subtracting x from each side, we get

$$3x-x-8=12.$$

Adding 8 to each side, we have

$$3x-x=12+8.$$

Thus we see that $+x$ has been removed from one side, and appears as $-x$ on the other; and -8 has been removed from one side and appears as $+8$ on the other.

Similar steps may be employed in all cases.

It appears from this that *we may change the sign of every term in an equation*; for this is equivalent to transposing all the terms, and then making the two sides change places.

Example. Take the equation $-3x-12=x-24$.

Transposing, $\quad -x+24=3x+12$,

or $\quad 3x+12=-x+24$,

which is the original equation with the sign of every term changed.

74. We can now give a general rule for solving any simple equation with one unknown quantity.

Rule. *First, if necessary, clear of fractions; then transpose all the terms containing the unknown quantity to one side of the equation, and the known quantities to the other. Collect the terms on each side; divide both sides by the coefficient of the unknown quantity and the value required is obtained.*

Example 1. Solve $5x-(4x-7)(3x-5)=6-3(4x-9)(x-1)$.

Here the products $(4x-7)(3x-5)$ and $(4x-9)(x-1)$ must be multiplied out, or written down by inspection as in Art. 44, before any further reduction can be made.

Forming the products, we have

$$5x-(12x^2-41x+35)=6-3(4x^2-13x+9);$$

and by removing brackets,

$$5x-12x^2+41x-35=6-12x^2+39x-27.$$

The term $-12x^2$ may be removed from each side without altering the equality; thus

$$5x+41x-35=6+39x-27.$$

Transposing, $\quad 5x+41x-39x=6-27+35;$

collecting terms, $\quad 7x=14;$

$$\therefore\ x=2.$$

Note. Since the minus sign before a bracket affects every term within it, in the first line of work we do not remove the brackets until we have formed the products.

Example 2. Solve $7x-5[x-\{7-6(x-3)\}]=3x+1$.

Removing brackets, we have

$$7x-5[x-\{7-6x+18\}]=3x+1,$$

$$7x-5[x-25+6x]=3x+1,$$

$$7x-5x+125-30x=3x+1;$$

transposing, $\quad 7x-5x-30x-3x=1-125;$

collecting terms, $\quad -31x=-124;$

$$\therefore\ x=4.$$

EXAMPLES VIII. a.

[*It is recommended that Nos.* 1–16 *of the following examples should be solved in full by reference to the axioms. In the rest of the exercise the solutions may be shortened by transposition of terms.*]

Solve the following equations and verify the solutions in Examples 1 to 20.

1. $3x+15=x+25$. **2.** $2x-3=3x-7$.

3. $3x+4=5(x-2)$. **4.** $2x+3=16-(2x-3)$.

5. $8(x-1)+17(x-3)=4(4x-9)+4$.

6. $15(x-1)+4(x+3)=2(7+x)$.

7. $5x-6(x-5)=2(x+5)+5(x-4)$.

8. $8(x-3)-(6-2x)=2(x+2)-5(5-x)$.

Solve the following equations:

9. $7(25-x)-2x=2(3x-25)$.
10. $3(169-x)-(78+x)=29x$.
11. $5x-17+3x-5=6x-7-8x+115$.
12. $7x-39-10x+15=100-33x+26$.
13. $118-65x-123=15x+35-120x$.
14. $157-21(x+3)=163-15(2x-5)$.
15. $179-18(x-10)=158-3(x-17)$.
16. $97-5(x+20)=111-8(x+3)$.
17. $x-[3+\{x-(3+x)\}]=5$.
18. $5x-(3x-7)-\{4-2x-(6x-3)\}=10$.
19. $14x-(5x-9)-\{4-3x-(2x-3)\}=30$.
20. $25x-19-[3-\{4x-5\}]=3x-(6x-5)$.
21. $(x+1)(2x+1)=(x+3)(2x+3)-14$.
22. $(x+1)^2-(x^2-1)=x(2x+1)-2(x+2)(x+1)+20$.
23. $2(x+1)(x+3)+8=(2x+1)(x+5)$.
24. $6(x^2-3x+2)-2(x^2-1)=4(x+1)(x+2)-24$.
25. $2(x-4)-(x^2+x-20)=4x^2-(5x+3)(x-4)-64$.
26. $(x+15)(x-3)-(x^2-6x+9)=30-15(x-1)$.
27. $2x-5\{3x-7(4x-9)\}=66$.
28. $20(2-x)+3(x-7)-2[x+9-3\{9-4(2-x)\}]=22$.
29. $x+2-[x-8-2\{8-3(5-x)-x\}]=0$.
30. $3(5-6x)-5[x-5\{1-3(x-5)\}]=23$.
31. $(x+1)(2x+3)=2(x+1)^2+8$.
32. $3(x-1)^2-3(x^2-1)=x-15$.
33. $(3x+1)(2x-7)=6(x-3)^2+7$.
34. $x^2-8x+25=x(x-4)-25(x-5)-16$.
35. $x(x+1)+(x+1)(x+2)=(x+2)(x+3)+x(x+4)-9$.
36. $2(x+2)(x-4)=x(2x+1)-21$.
37. $(x+1)^2+2(x+3)^2=3x(x+2)+35$.
38. $4(x+5)^2-(2x+1)^2=3(x-5)+180$.
39. $84+(x+4)(x-3)(x+5)=(x+1)(x+2)(x+3)$.
40. $(x+1)(x+2)(x+6)=x^3+9x^2+4(7x-1)$.

75. The following examples illustrate the most useful methods of solving equations with fractional coefficients.

Example 1. Solve $4 - \frac{x-9}{8} = \frac{x}{22} - \frac{1}{2}$.

Multiply by 88, the least common multiple of the denominators;

thus $\quad 352 - 11(x-9) = 4x - 44$;

removing brackets, $\quad 352 - 11x + 99 = 4x - 44$;

transposing, $\quad -11x - 4x = -44 - 352 - 99$;

collecting terms and changing signs, $\quad 15x = 495$;

$$\therefore \quad x = 33.$$

Note. Here $-\frac{x-9}{8}$ is equivalent to $-\frac{1}{8}(x-9)$, the *vinculum* or line between the numerator and denominator having the same effect as a bracket. [Art. 58.]

76. In certain cases it will be found more convenient not to multiply throughout by the L.C.M. of the denominator, but to clear of fractions in two or more steps.

Example 2. Solve $\frac{x-4}{3} + \frac{2x-3}{35} = \frac{5x-32}{9} - \frac{x+9}{28}$.

Multiplying throughout by 9, we have

$$3x - 12 + \frac{18x-27}{35} = 5x - 32 - \frac{9x+81}{28};$$

transposing, $\quad \frac{18x-27}{35} + \frac{9x+81}{28} = 2x - 20.$

Now clear of fractions by multiplying by $5 \times 7 \times 4$ or 140;

thus $\quad 72x - 108 + 45x + 405 = 280x - 2800$;

$$\therefore \quad 2800 - 108 + 405 = 280x - 72x - 45x;$$

$$\therefore \quad 3097 = 163x;$$

$$\therefore \quad x = 19.$$

77. To solve equations whose coefficients are decimals, we may express the decimals as vulgar fractions, and proceed as before; but it is often found more simple to work entirely in decimals.

Example 1. Solve $\cdot\dot{6}x + \cdot 25 - \frac{1}{9}x = 1\cdot\dot{8} - \cdot 75x - \frac{1}{3}$.

Expressing the decimals as vulgar fractions, we have

[illegible] $\quad \frac{2}{3}x + \frac{1}{4} - \frac{1}{9}x = 1\frac{8}{9} - \frac{3}{4}x - \frac{1}{3}$;

clearing of fractions, $\quad 24x + 9 - 4x = 68 - 27x - 12$;

transposing, $\quad 24x - 4x + 27x = 68 - 12 - 9$,

$$47x = 47;$$

$$\therefore \quad x = 1.$$

Example 2. Solve $\cdot375x - 1\cdot875 = \cdot12x + 1\cdot185$.

Transposing, $\cdot375x - \cdot12x = 1\cdot185 + 1\cdot875$;

collecting terms, $(\cdot375 - \cdot12)x = 3\cdot06$,

that is, $\cdot255x = 3\cdot06$;

$$\therefore\ x = \frac{3\cdot06}{\cdot255} = 12.$$

EXAMPLES VIII. b.

Solve the following equations, and verify Nos. 1–16.

1. $\frac{x}{4} + \frac{x-5}{3} = 10.$

2. $\frac{x-5}{10} + \frac{x+5}{5} = 5.$

3. $\frac{x-2}{2} + \frac{x+10}{9} = 5.$

4. $\frac{x+19}{5} = 3 + \frac{x}{4}.$

5. $\frac{x-4}{7} = \frac{x-10}{5}.$

6. $\frac{x-1}{8} = 1 + \frac{x+1}{18}.$

7. $\frac{4(x+2)}{5} = 7 + \frac{5x}{13}.$

8. $\frac{x+4}{14} + \frac{x-4}{6} = 2.$

9. $\frac{x+20}{9} + \frac{3x}{7} = 6.$

10. $\frac{x-8}{7} + \frac{x-3}{3} + \frac{5}{21} = 0.$

11. $\frac{x+5}{6} - \frac{x+1}{9} = \frac{x+3}{4}.$

12. $\frac{4-5x}{6} - \frac{1-2x}{3} = \frac{13}{42}.$

13. $\frac{5(x+5)}{8} - \frac{2(x-3)}{7} = 5\frac{19}{28}.$

14. $\frac{4(x+2)}{3} - \frac{6(x-7)}{7} = 12.$

15. $1 + \frac{x}{2} - \frac{2x}{3} = \frac{3x}{4} - 4\frac{1}{2}.$

16. $\frac{x}{2} + \frac{x}{3} - \frac{x}{4} + \frac{x}{5} = 7\frac{5}{6}.$

17. $\frac{3}{16}(x-1) - \frac{5}{12}(x-4) = \frac{2}{5}(x-6) + \frac{5}{48}.$

18. $x + \frac{5}{3}(x-7) - \frac{6}{7}(x-8) = 3x - 14\frac{1}{3}.$

19. $\frac{3x}{4} - \frac{6}{17}(x+10) - (x-3) = \frac{x-7}{51} - 4\frac{3}{4}.$

20. $\frac{7x}{5} - \frac{1}{14}(x-11) = \frac{3}{7}(x-25) + 34.$

21. $3 + \frac{x}{4} = \frac{1}{2}\left(4 - \frac{x}{3}\right) - \frac{5}{6} + \frac{1}{3}\left(11 - \frac{x}{2}\right).$

22. $\frac{1}{5}(x-8) + \frac{4+x}{4} + \frac{x-1}{7} = 7 - \frac{23-x}{5}.$

23. $\frac{1}{3}\left(\frac{x}{4}-3\right)+\frac{5x}{6}-\frac{5x}{4}=\frac{x-12}{5}-\frac{x+3}{3}$.

24. $x-\left(3x-\frac{2x-5}{10}\right)=\frac{1}{6}(2x-57)-\frac{5}{3}$.

25. $\frac{x}{4}-\frac{x+10}{5}+4\frac{3}{4}=x-1-\frac{x-2}{3}$.

26. $\cdot 5x-\cdot\dot{3}x=\cdot 25x-1$.

27. $3+\frac{x}{\cdot 5}=7-\frac{x}{\cdot 2}$.

28. $2\cdot 25x-\cdot 125=3x+3\cdot 75$.

29. $\cdot 2x-\cdot 1\dot{6}x=\cdot 6-\cdot\dot{3}$.

30. $\cdot 6x-\cdot 7x+\cdot 75x-\cdot 875x+15=0$.

31. $12\{3x-\cdot 25(x-4)-\cdot\dot{3}(5x+14)\}=47$.

32. $\frac{\cdot 25(x-3)+\cdot\dot{3}(x-4)}{\cdot 125}=5x-19$.

33. $\frac{x+\cdot 75}{\cdot 125}-\frac{x-\cdot 25}{\cdot 25}=15$.

34. $\frac{5}{6}x+\cdot 25x-\cdot\dot{3}x=x-3$.

35. $\cdot 5x-\cdot\dot{2}x=\cdot\dot{3}x-1\cdot 5$.

36. $1\cdot 5=\frac{\cdot 36}{\cdot 2}-\frac{\cdot 09x-\cdot 18}{\cdot 9}$.

[*Some of the examples in Miscellaneous Examples II., p.* 80, *will furnish further practice in Simple Equations.*]

78. Before concluding this chapter it will be worth while to draw attention to the following cases which occur so frequently in solving equations that the beginner should learn to write down the solution at sight.

Case I. Suppose
$$\frac{7x}{5}=\frac{4}{3}.$$

Multiplying both sides by 5, we have
$$\left.\begin{aligned} 7x&=\frac{4\times 5}{3} \\ \therefore\quad x&=\frac{4\times 5}{3\times 7}\end{aligned}\right\}\ \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Case II. Suppose
$$\frac{5}{3x}=\frac{9}{7}.$$

Multiplying both sides by $3x$, we have
$$\left.\begin{aligned} 5&=\frac{9\times 3x}{7} \\ 5\times 7&=9\times 3x \\ \frac{5\times 7}{9\times 3}&=x\end{aligned}\right\}\ \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

By a careful examination of the results in (1) and (2), the truth of the following principles will be evident:

Any factor of the numerator of one side of an equation may be transferred to the denominator of the other side, and any factor of the denominator of one side may be transferred to the numerator of the other side.

The ready application of these principles will be found very useful.

Example 1. If $$\frac{3x}{14}=\frac{9}{35},$$

then $$x=\frac{9\times 14}{35\times 3}=1\tfrac{1}{5}.$$

Example 2. If $$\frac{2}{x}=-5,$$

then $$\frac{2}{5}=-x; \qquad \therefore\ x=-\frac{2}{5}.$$

After a little practice the arithmetic should be performed mentally, and the intermediate steps omitted.

EXAMPLES VIII. c.

Write down the values of x which satisfy the following equations:

1. $\frac{2}{x}=\frac{3}{4}.$ 2. $\frac{3}{7}=\frac{x}{14}.$ 3. $\frac{3}{5}=\frac{6}{x}.$

4. $-\frac{x}{3}=2.$ 5. $\frac{x}{17}=\frac{29}{51}.$ 6. $\frac{2x}{15}=\frac{8}{45}.$

7. $\frac{3}{2x}=-\frac{1}{8}.$ 8. $-\frac{4x}{3}=-\frac{1}{2}.$ 9. $\frac{5}{3x}=\frac{25}{27}.$

10. $\frac{5}{2x}=-\frac{10}{3}.$ 11. $\frac{13}{21}=\frac{65x}{84}.$ 12. $-\frac{7}{2}=\frac{1}{3x}.$

13. $\frac{3}{8}=\frac{x}{4}.$ 14. $\frac{8}{21x}=-\frac{4}{7}.$ 15. $\frac{36}{35}=\frac{9}{5x}.$

16. $\frac{5}{8}=\frac{15}{2x}.$ 17. $\frac{x}{18}=\frac{9}{42}.$ 18. $\frac{4}{3x}=\frac{16}{27}.$

19. $\frac{49}{15}=\frac{7}{3x}.$ 20. $\frac{56}{15}=\frac{8x}{3}.$ 21. $\frac{19x}{7}=\frac{57}{49}.$

CHAPTER IX.

Symbolical Expression.

79. In solving algebraical problems the chief difficulty of the beginner is to express the conditions of the question by means of symbols. A question proposed in algebraical symbols will frequently be found puzzling, when a similar arithmetical question would present no difficulty. Thus, the answer to the question "find a number greater than x by a" may not be self-evident to the beginner, who would of course readily answer an analogous arithmetical question, "find a number greater than 50 by 6." The process of addition which gives the answer in the second case supplies the necessary hint; and, just as the number which is greater than 50 by 6 is $50+6$, so the number which is greater than x by a is $x+a$.

80. The following examples will perhaps be the best introduction to the subject of this chapter. After the first we leave to the student the choice of arithmetical instances, should he find them necessary.

Example 1. By how much does x exceed 17?

Take a numerical instance; "by how much does 27 exceed 17?"

The answer obviously is 10, which is equal to $27-17$.

Hence the excess of x over 17 is $x-17$.

Similarly the defect of x from 17 is $17-x$.

Example 2. If x is one *part* of 45 the other part is $45-x$.

Example 3. If x is one *factor* of 45 the other factor is $\frac{45}{x}$.

Example 4. How far can a man walk in a hours at the rate of 4 miles an hour?

In 1 hour he walks 4 miles,

In a hours he walks a times as far, that is, $4a$ miles.

Example 5. If \$20 is divided equally among y persons, the share of each is the total sum divided by the number of persons, or \$$\frac{20}{y}$.

Example 6. If 17 be divided by 6 the quotient is 2, and the remainder 5,

that is, $$\frac{17}{6}=2+\frac{5}{6}.$$

So if N be divided by D, and the quotient be Q and the remainder R, we have

$$\frac{N}{D}=Q+\frac{R}{D},$$

or $$N=QD+R.$$

Thus, if the divisor is x, the quotient y, and the remainder z, the dividend is $xy+z$.

Example 7. A has p quarters, and B has q ten-cent pieces; after B has received a sum equivalent to x dollars from A, how much money has each?

What B has received A has given away;

$$\therefore\ A \text{ has } \frac{p}{4}-x \text{ dollars,}$$

$$B \text{ has } \frac{q}{10}+x \text{ dollars.}$$

EXAMPLES IX. a.

1. What must be added to x to make y?

2. By what must 3 be multiplied to make a?

3. What dividend gives b as the quotient when 5 is the divisor?

4. What is the defect of $2c$ from $3d$?

5. By how much does $3k$ exceed k?

6. If 100 be divided into two parts and one part be x what is the other?

7. If a be one factor of b, what is the other?

8. What number is less than 20 by c?

9. What is the price in cents of a oranges at ten cents a dozen?

10. What is the price in dollars of 100 cakes when x cost ten cents?

11. If the difference of two numbers be 11, and if the smaller be x, what is the greater?

12. If the sum of two numbers be c and one of them is 20, what is the other?

13. What is the excess of 90 over x?

14. By how much does x exceed 30?

15. If 100 contains x five times, what is the value of x?

16. What is the cost in dollars of 40 books at x cents each?

17. In x years a man will be 36 years old, what is his present age?

18. How old will a man be in a years if his present age is x years?

19. If x men take 5 days to reap a field, how long will one man take?

20. What value of x will make $5x$ equal to 20?

21. What is the price in quarters of 120 apples, when the cost of a score is x cents?

22. How many hours will it take to walk x miles at 4 miles an hour?

23. How far can I walk in x hours at the rate of y miles an hour?

24. In x days a man walks y miles, what is his rate per day?

25. How many minutes will it take to walk x miles at a miles an hour?

26. A train goes x miles an hour, how long does it take to go from Toronto to London, a distance of 118 miles?

27. How many miles is it between two places, if a train travelling p miles an hour takes 5 hours to perform the journey?

28. What is the velocity in feet per second of a train which travels 30 miles in x hours?

29. A man has a dollars and b quarters, how many cents has he?

30. If I spend x half-dollars out of a sum of \$20, how many half-dollars have I left?

31. Out of a purse containing \$$a$ and b quarters a man spends c cents; express in cents the sum left.

32. By how much does $2x - 5$ exceed $x + 1$?

33. What number must be taken from $a - 2b$ to leave $a - 3b$?

34. If a bill is shared equally amongst x persons and each pays a quarter, how many cents does the bill amount to?

35. If I give away c quarters out of a purse containing a half-dollars and b dollars, how many quarters have I left?

36. In how many weeks will x horses eat 100 bushels of oats if one horse eats y bushels a week?

37. If I spend x quarters a week, how many dollars do I save out of a yearly income of \$$y$?

38. A bookshelf contains x Latin, y Greek, and z English books: if there are 100 books, how many are there in other languages?

39. I have x dollars in my purse, y ten-cent pieces in one pocket, and z cents in another: if I give away a half-dollar, how many cents have I left?

40. In a class of x boys, y work at Classics, z at Mathematics, and the rest are idle: what is the excess of workers over idlers?

81. We subjoin a few harder examples worked out in full.

Example 1. What is the present age of a man who x years hence will be m times as old as his son now aged y years?

In x years the son's age will be $y+x$ years; hence the father's age will be $m(y+x)$ years; therefore *now* the father's age is $m(y+x)-x$ years.

Example 2. Find the simple interest on \$$k$ in n years at f per cent.

Interest on \$100 for 1 year is \$$f$,

$\therefore$ .... \$1 .... \$$\frac{f}{100}$,

$\therefore$ .... \$$k$ .... \$$\frac{kf}{100}$,

$\therefore$ Interest on \$$k$ for n years is \$$\frac{nkf}{100}$.

Example 3. A room is x yards long, y feet broad, and a feet high; find how many square yards of carpet will be required for the floor, and how many square yards of paper for the walls.

(1) The area of the floor is $3xy$ square feet;

$\therefore$ the number of square yards of carpet required is $\frac{3xy}{9}=\frac{xy}{3}$.

(2) The perimeter of the room is $2(3x+y)$ feet;

$\therefore$ the area of the walls is $2a(3x+y)$ square feet;

$\therefore$ number of square yards of paper required is $\frac{2a(3x+y)}{9}$.

Example 4. The digits of a number beginning from the left are a, b, c; what is the number?

Here c is the digit in the units' place; b standing in the tens' place represents b tens; similarly a represents a hundreds.
The number is therefore equal to a hundreds $+\,b$ tens $+\,c$ units

$$=100a+10b+c.$$

If the digits of the number are inverted, a new number is formed which is symbolically expressed by

$$100c+10b+a.$$

Example 5. What is (1) the sum, (2) the product of three consecutive numbers of which the least is n?

The numbers consecutive to n are $n+1$, $n+2$;

$$\therefore \quad \text{the sum} = n+(n+1)+(n+2)$$
$$=3n+3.$$

And the product $=n(n+1)(n+2)$.

We may remark here that any *even* number may be denoted by $2n$, where n is *any* positive whole number; for this expression is exactly divisible by 2.

Similarly, any odd number may be denoted by $2n+1$; for this expression when divided by 2 leaves remainder 1.

Example 6. How many days will a men take to mow b acres if c boys can mow a acres in b days, and each man's work equals that of n boys?

Since c boys can mow a acres in b days;

$\therefore$ 1 boy.................................... bc days,

$\therefore$ n boys, or 1 man, ... $\frac{bc}{n}$ days,

$\therefore$ a men.................................. $\frac{bc}{an}$ days,

$\therefore$ a men...................... 1 acre... $\frac{bc}{a^2n}$ days;

therefore a men can mow b acres in $\frac{b^2c}{a^2n}$ days.

EXAMPLES IX. b.

1. Write down four consecutive numbers of which x is the least.

2. Write down three consecutive numbers of which y is the greatest.

3. Write down five consecutive numbers of which x is the middle one.

4. What is the next even number after $2n$?

5. What is the odd number next before $2x+1$?

6. Find the sum of three consecutive odd numbers of which the middle one is $2n+1$.

7. A man makes a journey of x miles. He travels a miles by coach, b by train, and finishes the journey by boat. How far does the boat carry him?

8. A horse eats a bushels and a donkey b bushels of corn in a week: how many bushels will they together consume in n weeks?

9. If a man was x years old 5 years ago, how old will he be y years hence?

10. A boy is x years old, and 5 years hence his age will be half that of his father. How old is the father now?

11. What is the age of a man who y years ago was m times as old as a child then aged x years?

12. A's age is double B's, B's is three times C's, and C is x years old: find A's age.

13. What is the interest on \$1000 in b years at c per cent.?

14. What is the interest on $\$x$ in a years at 5 per cent.?

15. What is the interest on $\$50a$ in a years at a per cent.?

16. What is the interest on $\$24xy$ in x months at y per cent. per annum?

17. A room is x yards in length, and y feet in breadth: how many square feet are there in the area of the floor?

18. A square room measures x feet each way: how many square yards of carpet will be required to cover it?

19. A room is p feet long and x yards in width: how many yards of carpet two feet wide will be required for the floor?

20. What is the cost in dollars of carpeting a room a yards long, b feet broad, with carpet costing c cents a square yard?

21. How many yards of carpet x inches wide will be required to cover the floor of a room y feet long and z feet broad?

22. A room is a yards long and b yards broad; in the middle there is a carpet c feet square: how many square yards of oil-cloth will be required to cover the rest of the floor?

23. How many miles can a person walk in 45 minutes if he walks a miles in x hours?

24. How long will it take a person to walk b miles if he walks 20 miles in c hours?

25. If a train travels a miles in b hours, how many feet does it move through in one second?

26. A train is running with a velocity of x feet per second: how many miles will it travel in y hours?

27. How long will x men take to mow y acres of corn, if each man mow z acres a day?

28. How many men will be required to do in x hours what y men do in xz hours?

29. What is the rate per cent. which will produce $\$y$ interest from a principal of \$1000 in r years?

30. Find in how many years a principal of $\$a$ will produce $\$p$ interest at r per cent. per annum.

[*The following examples will assist the student in stating the conditions of a problem in equational form.*]

31. If y is the product of three consecutive numbers of which the greatest is p, express this fact by an equation.

32. The sum of three consecutive even numbers is equal to x. If the middle number is $2n$ express this by an equation.

33. The product of p and q is equal to five times the excess of a over b; express this by an equation.

34. If x is divided by y, the quotient is equal to 10 more than the sum of m and n; express this in algebraical symbols.

35. A man is x years older than his son, whose present age is a years; five years hence the father's age will be twice that of the son; express this in algebraical symbols. If the son is now 15, what is the father's age? If the father is now 53, how old is the son?

36. A has \$$p$ and B has q cents; A hands \$$x$ to B and finds that he then has three times as much as B; express this fact by an equation.

37. A man who is p years old has a son whose age is q years; five years ago the father's age was seven times that of his son. Express this in algebraical symbols.

Formulæ.

82. In Example 6, Art. 80, we proved

$$\frac{N}{D}=Q+\frac{R}{D},$$

a result which gives in a single statement a general relation expressing the connection between a number, its divisor, and resulting quotient and remainder.

This is an example of a very important class of algebraical statements known as *formulæ*, the use and application of which we shall now briefly explain.

Definition. A **formula** is a relation established by reasoning among certain quantities, any one of which may in turn be regarded as the unknown.

Thus in the formula above mentioned, if Q, R, and D are given quantities, we have an equation to find the corresponding value of N. Or, a question may be proposed as follows: "By what must 96 be divided so as to give a quotient 5, and a remainder 11?" Here we have given $N=96$, $Q=5$, $R=11$, and therefore from the formula we obtain

$$\frac{96}{D}=5+\frac{11}{D},$$

whence $D=17$, the required divisor.

83. A formula, it must be observed, includes all particular cases in one general statement; and so by the use of a single algebraical formula we are enabled briefly to express a whole class of results in a form at once simple, easily remembered, and easily applied. Experience will convince the student how much of the power and utility of Algebra lies in the ready application of formulæ to many kinds of problems.

It would be out of place here to make more than a passing allusion to other branches of Mathematics, or to Physical Science; but on account of the interest and importance of the subject, it may be useful to draw the reader's attention to a few of the more elementary formulæ he is likely to meet with in his other studies.

(1) If a triangle on a base b, has a height h, its area (A) is given by the formula $A=\frac{1}{2}hb$.

(2) If a pyramid of height h stands on a base whose area is a^2, its volume (V) is given by the formula

$$V=\tfrac{1}{3}a^2h.$$

In these cases any linear unit, inch, foot ... being chosen, the superficial and solid units will be respectively the square and cubic inch, foot, ... ; and in each of these formulæ if two of the three quantities be given, the third is easily obtained by Arithmetic.

Example. The Great Pyramid of Egypt stands on a square base each side of which is 764 feet; and its height is 480 feet. Find the number of cubic feet of stone used in its construction.

From the formula, $V=\frac{1}{3}\times(764)^2\times 480$
$=160\times 764\times 764$
$=93391360$ cubic feet.

84. We have in this chapter given several examples involving space, velocity, and time; and all these can be solved without difficulty by common sense reasoning. At the same time we may remark that they are only particular cases of the general formula $s=vt$, in which s denotes the space described by a body which moves with uniform velocity v for a time t.

In this formula, if t denotes the number of seconds the body has been in motion, and v the number of feet passed over in one second, then s is the space (in feet) described in t seconds.

Example. If a train has a velocity of 75 feet a second, how long will it take to cross a viaduct which is 300 yards in length?

Substituting the values of s and v (expressed in feet) in the formula, we get

$$900=75t,$$
$$t=\frac{900}{75}$$
$$=12.$$

Therefore the time is 12 seconds.

85. Another very interesting case is that of a body falling vertically under the action of gravity.

It is proved in works on Dynamics that if a body fall freely from rest, and if s denote the space (in feet) described in t seconds,

$$s = \tfrac{1}{2}gt^2.$$

In this formula g denotes the number of feet per second by which the velocity is increased in each successive second in consequence of the earth's attraction, and it is found by experiment that $g = 32{\cdot}2$ nearly.

Example 1. A stone dropped from the Clifton suspension bridge takes 4 seconds before it reaches the water. Find the height of the bridge above the river.

From the above formula, $s = \frac{1}{2} \times 32{\cdot}2 \times (4)^2$

$$= 257{\cdot}6,$$

and the height is therefore 257·6 feet.

Example 2. How long will it take a stone to reach the bottom of a well 144·9 feet deep?

From the formula, $144{\cdot}9 = \frac{1}{2} \times 32{\cdot}2 \times t^2$;

$$\therefore\ t^2 = \frac{144{\cdot}9}{16{\cdot}1} = 9;$$

$$\therefore\ t = 3.$$

Therefore the time is 3 seconds.

EXAMPLES IX. c.

1. From the formula for the area of a triangle in Art. 83, find

(i) The area, when the base is 32 ft., and the height 17 ft.

(ii) The base, when the area is 56 sq. ft., and the height 7 ft.

(iii) The height (in chains and links), when the area is 5·985 acres, and the base 17 chains 50 links.

2. By means of formula (2) in Art. 83, find

(i) The volume of a pyramid of height 10 ft., on a base whose area is 15 sq. ft.

(ii) The volume of a pyramid of height 6 ft., standing on a square base each of whose sides is $1\frac{1}{2}$ ft.

(iii) The height of a pyramid whose volume is 20 cu. ft. and whose base has an area of 12 sq. ft.

3. By means of the formula $s=vt$ (Art. 84), find

(i) How many miles a train will run in 84 minutes at 35 miles per hour.

(ii) How long a train will take to run 56 miles at 42 miles per hour.

(iii) The velocity in miles per hour of a train which travels 5500 yards in 5 minutes.

4. By means of the formula $s=\frac{1}{2}gt^2$ (Art. 85), find

(i) The height of a flagstaff if a stone dropped from the top takes 3 seconds to reach the ground.

(ii) How long it will take a stone to drop from a balloon whose height above the ground is 402 ft. 6 in.

5. The circumference (C) of a circle is π times the diameter (d); and the area (A) of a circle is π times the square of the radius (r). Express these two results by formulæ.

If $\pi=\frac{22}{7}$, find the circumferences and areas of circles whose radii are $3\frac{1}{2}$ inches and 1 ft. 9 in. respectively.

6. The surface S of a sphere of radius r is given by the formula

$$S=4\times\frac{22}{7}r^2.$$

Find (i) the surface of a sphere whose radius is 1·4 in.;

(ii) the radius of a sphere whose surface is $38\frac{1}{2}$ sq. ft.

7. If a room is x feet long, y feet broad, and z feet high, find formulæ for (i) the perimeter, (ii) the area of the floor, (iii) the area of the walls.

8. From the formulæ of the last example find the perimeter, area of floor, and area of the walls of a room 18 ft. 8 in. long, 11 ft. 3 in. wide, and 12 ft. high.

9. From formula (iii) of Example 7, find the height of a room when the length and breadth are 17 ft. 9 in., 12 ft. 3 in. respectively, and the area of the walls is 630 sq. ft.

10. If a parallelogram on a base b has a height h, its area (A) is given by the formula

$$A=bh.$$

Find the area of parallelograms in which

(i) the base $=5{\cdot}5$ cm., and the height $=4$ cm.;

(ii) the base $=2{\cdot}4$ in., and the height $=1{\cdot}5$ in.

11. The area of a parallelogram is 4·2 sq. in., and the base is 2·8 in. Find the height.

12. The area of a trapezium is equal to

$\frac{1}{2}$(sum of parallel sides) × (distance between them).

Express this in algebraical symbols, and apply the formula to find the area of a trapezium when the parallel sides are 6 ft. 4 in. and 7 ft. 2 in. and the distance between them is 4 ft.

13. Use the formula of Art. 80, Ex. 6, to find a number which when divided by 19 gives a quotient 17 and remainder 5.

14. By what number must 566 be divided so as to give a quotient 37 and remainder 11?

15. What is the present age of a man who 5 years hence will be three times as old as his son who is now 15? Verify the answer by substituting in the formula of Art. 81, Ex. 1.

16. In a right-angled triangle if a and b denote the lengths of the sides containing the right angle and c denotes the length of the hypotenuse, it is known that $c^2 = a^2 + b^2$.

By substitution find which of the following sets of numbers can be taken to represent the sides of a right-angled triangle.

(i) 7, 24, 25. (ii) 12, 35, 36. (iii) 1·6, 6·3, 6·5.

17. The rectangle contained by two straight lines, one of which is divided into any number of parts, is equal to the sum of the rectangles contained by the undivided line and the several parts of the divided line.

Prove this by taking algebraical symbols to represent the undivided line and the segments of the divided line.

18. AB is a straight line divided into any two parts at O. Prove algebraically, as in the last example:

(i) $AB^2 = AB \,.\, AO + AB \,.\, OB.$

(ii) $AB \,.\, AO = AO^2 + AO \,.\, OB.$

Express these two results in a verbal form as in Example 17.

19. Prove algebraically the following theorems:

(i) If a straight line is divided into any two parts, the square on the whole line is equal to the sum of the squares on the two parts together with twice the rectangle contained by the two parts.

(ii) If a straight line is divided into any two parts, the sum of the squares on the whole line and on one of the parts is equal to twice the rectangle contained by the whole and that part, together with the square on the other part.

Express the results of these theorems in a form corresponding to (i) and (ii) of Example 18.

20. With the notation of Example 16, find the value of

(i) c when $a=15$, $b=8$;

(ii) a when $c=25$, $b=7$;

(iii) b when $c=41$, $a=9$;

(iv) a when $c=6{\cdot}5$, $b=6{\cdot}3$.

21. If $\pi=3{\cdot}1416$, $l=2{\cdot}0125$, $s=144{\cdot}9$, $g=32{\cdot}2$, $m=18{\cdot}75$, $v=5{\cdot}6$, find the values of

(i) $\pi\sqrt{\frac{l}{g}}$; (ii) $\sqrt{2gs}$; (iii) $\frac{1}{2}mv^2$.

22. In the formula $F=\frac{mv^2}{gr}$, given $m=12{\cdot}075$, $r=3$, $g=32{\cdot}2$, $F=200$, find v.

23. In the formula $v^2-u^2=2as$, find the value of a when $v=50$, $u=10$, and $s=100$.

24. From the formula $s=\frac{n}{2}(a+l)$, find

(i) the value of s, when $n=20$, $a=14$, $l=964$;

(ii) the value of a, when $s=25{\cdot}2$, $n=12$, $l=3{\cdot}2$;

(iii) the value of n, when $s=46{\cdot}8$, $a={\cdot}6$, $l=7{\cdot}2$;

(iv) the value of l, when $s=-175{\cdot}5$, $a=13{\cdot}5$, $n=13$.

25. If $y=4+\frac{3}{10}x$, find the value of y when x has the values 0, 4, 8, 12, 16, 20.

There is a wall 20 ft. long, whose height at any point x ft. from one end is $4+\frac{3}{10}x$ feet. Draw the wall on a scale of 1 inch to 4 feet, marking on it the height at each end and at intervals of 4 ft.

CHAPTER X.

PROBLEMS LEADING TO SIMPLE EQUATIONS.

86. THE principles of the last chapter may now be employed to solve various problems.

The method of procedure is as follows :

Represent the unknown quantity by a symbol x, and express in symbolical language the conditions of the question ; we thus obtain a simple equation which can be solved by the methods already given in Chapter VIII.

Example 1. Find two numbers whose sum is 28, and whose difference is 4.

Let x be the smaller number, then $x+4$ is the greater.

Their sum is $x+(x+4)$, which is to be equal to 28.

Hence
$$x+x+4=28;$$
$$2x=24;$$
$$\therefore \quad x=12,$$
and
$$x+4=16;$$
so that the numbers are 12 and 16.

The beginner is advised to test his solution by finding whether it satisfies the data of the question or not.

Example 2. Divide 60 into two parts, so that three times the greater may exceed 100 by as much as 8 times the less falls short of 200.

Let x be the greater part, then $60-x$ is the less.

Three times the greater part is $3x$, and its excess over 100 is
$$3x-100.$$

Eight times the less is $8(60-x)$, and its defect from 200 is
$$200-8(60-x).$$

Whence the symbolical statement of the question is
$$3x-100=200-8(60-x);$$
$$3x-100=200-480+8x,$$
$$480-100-200=8x-3x,$$
$$5x=180,$$
$$\therefore \quad x=36, \text{ the greater part,}$$
and
$$60-x=24, \text{ the less.}$$

Example 3. Divide \$47 between A, B, C, so that A may have \$10 more than B, and B \$8 more than C.

Suppose that C has x dollars; then B has $x+8$ dollars, and A has $x+8+10$ dollars.

Hence $$x+(x+8)+(x+8+10)=47;$$
$$x+x+8+x+8+10=47,$$
$$3x=21;$$
$$\therefore\ x=7;$$
so that C has \$7, B \$15, A \$25.

Example 4. A person spent \$112.80 in buying geese and ducks; if each goose cost \$1.40, and each duck 60 cents, and if the total number of birds bought was 108, how many of each did he buy?

In questions of this kind it is of essential importance to have all quantities expressed in the same denomination; in the present instance it will be convenient to express the money in cents.

Let x represent the number of geese, then $108-x$ represents the number of ducks.

Since each goose cost \$1.40, x geese cost $140\ x$ cents.

And since each duck cost 60 cents, $108-x$ ducks cost $60(108-x)$ cents.

Therefore the amount spent is
$$140x+60(108-x) \text{ cents};$$
but the question states that the amount is also \$112.80, that is, 11280 cents.

Hence $$140x+60(108-x)=11280;$$
dividing by 20, $$7x+324-3x=564,$$
$$4x=240;$$
$$\therefore\ x=60, \text{ the number of geese};$$
and $$108-x=48, \text{ the number of ducks}.$$

Example 5. A is twice as old as B, ten years ago he was four times as old: what are their present ages?

Let B's age be x years, then A's age is $2x$ years

Ten years ago their ages were respectively, $x-10$ and $2x-10$ years; thus we have $$2x-10=4(x-10);$$
$$2x-10=4x-40,$$
$$2x=30;$$
$$\therefore\ x=15,$$
so that B is 15 years old, A 30 years.

Note. In the above examples the unknown quantity x represents a *number* of dollars, ducks, years, etc.; and the student must be careful to avoid beginning a solution with a supposition of the kind, "let $x=A$'s share" or "let $x=$the ducks", or any statement so vague and inexact.

EXAMPLES X. a.

1. One number exceeds another by 5, and their sum is 29; find them.

2. The difference between two numbers is 8; if 2 be added to the greater the result will be three times the smaller: find the numbers.

3. Find a number such that its excess over 50 may be greater by 11 than its defect from 89.

4. A man walks 10 miles, then travels a certain distance by train, and then twice as far by coach. If the whole journey is 70 miles, how far does he travel by train?

5. What two numbers are those whose sum is 58, and difference 28?

6. If 288 be added to a certain number, the result will be equal to three times the excess of the number over 12: find the number.

7. Twenty-three times a certain number is as much above 14 as 16 is above seven times the number: find it.

8. Divide 105 into two parts, one of which diminished by 20 shall be equal to the other diminished by 15.

9. Find three consecutive numbers whose sum shall equal 84.

10. The sum of two numbers is 8, and one of them with 22 added to it is five times the other: find the numbers.

11. Find two numbers differing by 10 whose sum is equal to twice their difference.

12. A and B each have \$60. A gift from B makes A's money double B's, what does A receive?

13. Find a number such that if 5, 15, and 35 are added to it, the product of the first and third results may be equal to the square of the second.

14. The difference between the squares of two consecutive numbers is 121: find the numbers.

15. The difference of two numbers is 3, and the difference of their squares is 27: find the numbers.

16. Divide \$380 between A, B, and C, so that B may have \$30 more than A, and C may have \$20 more than B.

17. A sum of \$7 is made up of 46 coins which are either quarters or ten-cent pieces: how many are there of each?

18. If silk costs five times as much as linen, and I spend \$48 in buying 22 yards of silk and 50 yards of linen, find the cost of each per yard.

19. A father is four times as old as his son; in 24 years he will only be twice as old: find their ages.

20. A is 25 years older than B, and A's age is as much above 20 as B's is below 85: find their ages.

21. A's age is six times B's, and fifteen years hence A will be three times as old as B: find their ages.

22. A sum of \$16 was paid in dollars, half-dollars, and ten-cent pieces. The number of half-dollars used was four times the number of dollars and twice the number of ten-cent pieces; how many were there of each?

23. The sum of the ages of A and B is 30 years, and five years hence A will be three times as old as B: find their present ages.

24. In a cricket match the byes were double of the wides, and the remainder of the score was greater by three than twelve times the number of byes. If the whole score was 138, how were the runs obtained?

25. The length of a room exceeds its breadth by 3 feet; if the length had been increased by 3 feet, and the breadth diminished by 2 feet, the area would not have been altered: find the dimensions.

26. The length of a room exceeds its breadth by 8 feet; if each had been increased by 2 feet, the area would have been increased by 60 square feet: find the original dimensions of the room.

87. We add some problems which lead to equations with fractional coefficients.

Example 1. Find two numbers which differ by 4, and such that one-half of the greater exceeds one-sixth of the less by 8.

Let x be the smaller number, then $x+4$ is the greater.

One-half of the greater is represented by $\frac{1}{2}(x+4)$, and one sixth of the less by $\frac{1}{6}x$.

Hence $\frac{1}{2}(x+4)-\frac{1}{6}x=8$;

multiplying by 6, $3x+12-x=48$;

$\therefore\ 2x=36$;

$\therefore\ x=18$, the less number,

and $x+4=22$, the greater.

Example 2. A has \$180, and B has \$84: after B has received from A a certain sum, A has then five-sixths of what B has; how much did B receive?

Suppose that B receives x dollars, A has then $180-x$ dollars, and B has $84+x$ dollars.

Hence $180-x=\frac{5}{6}(84+x)$;

$$1080-6x=420+5x,$$

$$11x=660;$$

$$\therefore\ x=60.$$

Therefore B receives \$60.

EXAMPLES X. b.

1. Find a number such that the sum of its sixth and ninth parts may be equal to 15.

2. What is the number whose eighth, sixth, and fourth parts together make up 13?

3. There is a number whose fifth part is less than its fourth part by 3: find it.

4. Find a number such that six-sevenths of it shall exceed four-fifths of it by 2.

5. The fifth, fifteenth, and twenty-fifth parts of a number together make up 23: find the number.

6. Two consecutive numbers are such that one-fourth of the less exceeds one fifth of the greater by 1: find the numbers.

7. Two numbers differ by 28, and one is eight-ninths of the other: find them.

8. There are two consecutive numbers such that one-fifth of the greater exceeds one-seventh of the less by 3: find them.

9. Find three consecutive numbers such that if they be divided by 10, 17, and 26 respectively, the sum of the quotients will be 10.

10. A and B have equal sums of money; if B pays A five-elevenths of what he had at first, A would now have \$6 more than half of what B has left: what had they at first?

11. From a certain number 3 is taken, and the remainder is divided by 4; the quotient is then increased by 4 and divided by 5 and the result is 2: find the number.

12. In a cellar one-fifth of the wine is port and one-third claret; besides this it contains 15 dozen of sherry and 30 bottles of hock: how much port and claret does it contain?

13. Two-fifths of A's money is equal to B's, and seven-ninths of B's is equal to C's; in all they have \$770: what have they each?

14. A, B, and C have \$1285 between them: A's share is greater than five-sixths of B's by \$25, and C's is four-fifteenths of B's: find the share of each.

15. A man sold a horse for \$35 and half as much as he gave for it, and gained thereby ten dollars: what did he pay for the horse?

16. The width of a room is two-thirds of its length. If the width had been 3 feet more, and the length 3 feet less, the room would have been square: find its dimensions.

17. What is the property of a person whose income is \$430, when he has two-thirds of it invested at 4 per cent., one-fourth at 3 per cent., and the remainder at 2 per cent.?

18. I bought a certain number of apples at three for a cent, and five-sixths of that number at four for a cent: by selling them at sixteen for six cents I gain $3\frac{1}{2}$ cents: how many apples did I buy?

CHAPTER XI.

Highest Common Factor, Lowest Common Multiple of Simple Expressions.

Highest Common Factor.

88. Definition. The **highest common factor** of two or more algebraical expressions is the expression of highest dimensions [Art. 24] which divides each of them without remainder.

The abbreviation H.C.F. is sometimes used instead of the words *highest common factor*.

89. In the case of *simple expressions* the highest common factor can be written down by inspection.

Example 1. The highest common factor of a^4, a^3, a^2, a^6 is a^2.

Example 2. The highest common factor of a^3b^4, ab^5c^2, a^2b^7c is ab^4; for a is the highest power of a that will divide a^3, a, a^2; b^4 is the highest power of b that will divide b^4, b^5, b^7; and c is not a *common* factor.

90. If the expressions have numerical coefficients, find by Arithmetic their greatest common measure, and prefix it as a coefficient to the algebraical highest common factor.

Example. The highest common factor of $21a^4x^3y$, $35a^2x^4y$, $28a^3xy^4$ is $7a^2xy$; for it consists of the product of

(1) the greatest common measure of the numerical coefficients;

(2) the highest power of each letter which divides every one of the given expressions.

EXAMPLES XI. a.

Find the highest common factor of

1. $4ab^2$, $2a^2b$. **2.** $3x^2y^2$, x^3y^2. **3.** $6xy^2z$, $8x^2y^3z^2$.

4. abc, $2ab^2c$. **5.** $5a^3b^3$, $15abc^2$. **6.** $9x^2y^2z^2$, $12xy^3z$.

7. $4a^2b^3c^2$, $6a^3b^2c^3$. **8.** $7a^2b^4c^5$, $14ab^2c^3$.

9. $15x^4y^3z^2$, $12x^2yz^2$. **10.** $8a^2x$, $6abxy$, $10abx^3y^2$.

11. $49ax^2$, $63ay^2$, $56az^2$. **12.** $17ab^2c$, $34a^2bc$, $51abc^2$.

13. $a^3x^2y^2$, b^3xy^2, c^3x^2y. **14.** $24a^2b^3c^3$, $64a^3b^3c^2$, $48a^3b^2c^3$.

15. $25xy^2z$, $100x^2yz$, $125xy$. **16.** a^2bpxy, b^2qxy, a^3bxr^2.

17. $15a^5b^3c^7$, $60a^3b^7c^6$, $25a^4b^5c^2$. **18.** $35a^2c^3b$, $42a^3cb^2$, $30ac^2b^3$.

Lowest Common Multiple.

91. DEFINITION. The **lowest common multiple** of two or more algebraical expressions is the expression of lowest dimensions which is divisible by each of them without remainder.

The abbreviation L.C.M. is sometimes used instead of the words *lowest common multiple.*

92. In the case of *simple expressions* the lowest common multiple can be written down by inspection.

Example 1. The lowest common multiple of a^4, a^3, a^2, a^6 is a^6.

Example 2. The lowest common multiple of a^3b^4, ab^5, a^2b^7 is a^3b^7; for a^3 is the lowest power of a that is divisible by each of the quantities a^3, a, a^2; and b^7 is the lowest power of b that is divisible by each of the quantities b^4, b^5, b^7.

93. If the expressions have numerical coefficients, find by Arithmetic their least common multiple, and prefix it as a coefficient to the algebraical lowest common multiple.

Example. The lowest common multiple of $21a^4x^3y$, $35a^2x^4y$, $28a^3xy^4$ is $420a^4x^4y^4$; for it consists of the product of

(1) the least common multiple of the numerical coefficients;

(2) the lowest power of each letter which is divisible by every power of that letter occurring in the given expressions.

EXAMPLES XI. b.

Find the lowest common multiple of

1. abc, $2a^2$. **2.** x^3y^2, xyz. **3.** $3x^2yz$, $4x^3y^3$.

4. $5a^2bc^3$, $4ab^2c$. **5.** $3a^4b^2c^3$, $5a^2b^3c^5$. **6.** $12ab$, $8xy$.

7. ac, bc, ab. **8.** a^2c, bc^2, cb^2. **9.** $2ab$, $3bc$, $4ca$.

10. $2x$, $3y$, $4z$. **11.** $3x^2$, $4y^2$, $3z^2$. **12.** $7a^2$, $2ab$, $3b^3$.

13. a^2bc, b^2ca, c^2ab. **14.** $5a^2c$, $6cb^2$, $3bc^2$.

15. $2x^2y^3$, $3xy$, $4x^3y^4$. **16.** $7x^4y$, $8xy^5$, $2x^3y^3$.

17. $35a^2c^3b$, $42a^3cb^2$, $30ac^2b^3$. **18.** $66a^4b^2c^3$, $44a^3b^4c^2$, $24a^2b^3c^4$.

Find both the highest common factor and the lowest common multiple of

19. $2abc$, $3ca$, $4bca$. **20.** $2xy$, $4yz$, $6zxy$. **21.** $9abc$, $3b^2c$, cab.

22. $13a^2bc$, $39a^3bc^2$. **23.** $17xyz^2$, $51x^2y$. **24.** $15x^3y^3z$, $25xy^3z^2$.

25. $3ab$, $2bc$, $5cab$. **26.** $17m^2n^4p^2$, $51m^4p^4$. **27.** x^3y^2, y^2z^4, $z^4x^3y^5$.

28. $15p^3q^4$, $20m^2p^2q^3$, $30mp^3$. **29.** $72k^2m^3n^4$, $108k^3m^2n^5$.

CHAPTER XII.

ELEMENTARY FRACTIONS.

94. DEFINITION. If a quantity x be divided into b equal parts, and a of these parts be taken, the result is called *the fraction* $\frac{a}{b}$ *of* x. If x be the unit, the fraction $\frac{a}{b}$ of x is called simply "the fraction $\frac{a}{b}$"; so that *the fraction* $\frac{a}{b}$ *represents* a *equal parts,* b *of which make up the unit.*

95. In this chapter we propose to deal only with the easier kinds of fractions, where the numerator and denominator are simple expressions. Their reduction and simplification will be performed by the usual arithmetical rules. The proofs of these rules will be given in Chapters XIX. and XXI.

Rule. To reduce a fraction to its lowest terms: *divide numerator and denominator by every factor which is common to them both, that is by their highest common factor.*

Dividing numerator and denominator of a fraction by a common factor is called *cancelling* that factor.

Examples. (1) $\dfrac{6a^2c}{9ac^2}=\dfrac{2a}{3c}$.

(2) $\dfrac{7x^2yz}{28x^3yz^2}=\dfrac{1}{4xz}$.

(3) $\dfrac{35a^5b^3c}{7ab^2c}=\dfrac{5a^4b}{1}=5a^4b$.

EXAMPLES XII. a.

Reduce to lowest terms:

1. $\dfrac{3a}{6ab}$. **2.** $\dfrac{4a^2}{16ab}$. **3.** $\dfrac{2xy^2}{5x^2y}$. **4.** $\dfrac{3abc}{15a^2b^2c}$.

5. $\dfrac{x^2yz^3}{x^3y^2z}$. **6.** $\dfrac{15ab}{25bc}$. **7.** $\dfrac{21x^2y^2}{28y^2z^2}$. **8.** $\dfrac{8a^2b}{12b^2c}$.

9. $\dfrac{12mn^2p}{15m^2np^2}$. **10.** $\dfrac{15m^2p^3}{18n^4p}$. **11.** $\dfrac{abc^2}{a^3b^2c}$. **12.** $\dfrac{3x^2yz^3}{5xy^4z^2}$.

13. $\dfrac{2xy^3z^4}{4x^2y^2z}$. **14.** $\dfrac{5a^3b^2c^4}{15ab^4c}$. **15.** $\dfrac{mn^4pq}{m^2n^3p^4}$. **16.** $\dfrac{4m^3n^2p^5}{6m^4np^2}$.

17. $\dfrac{15ax^3y^2}{25a^2xy^6}$. **18.** $\dfrac{39a^2b^4c^3}{52a^3b^5c^4}$. **19.** $\dfrac{38k^2p^3m^4}{57k^3pm^2}$. **20.** $\dfrac{46x^3y^4z^5}{69x^2y^3z^4}$.

Multiplication and Division of Fractions.

96. Rule. To multiply algebraical fractions: *as in Arithmetic, multiply together all the numerators to form a new numerator, and all the denominators to form a new denominator.*

Example 1. $\frac{2a}{3b} \times \frac{5x^2}{2a^2b} \times \frac{3b^2}{2x} = \frac{2a \times 5x^2 \times 3b^2}{3b \times 2a^2b \times 2x} = \frac{5x}{2a}$,

by cancelling like factors in numerator and denominator.

Example 2. $\frac{3a^2b}{5c^2} \times \frac{7bc}{3a^3} \times \frac{5ca}{7b^2} = 1$,

all the factors cancelling each other.

97. Rule. To divide one fraction by another: *invert the divisor and proceed as in multiplication.*

Example. $\frac{7a^3}{4x^3y^2} \times \frac{6c^3x}{5ab^2} \div \frac{28a^2c^2}{15b^2xy^2} = \frac{7a^3}{4x^3y^2} \times \frac{6c^3x}{5ab^2} \times \frac{15b^2xy^2}{28a^2c^2} = \frac{9c}{8x}$,

all the other factors cancelling each other.

EXAMPLES XII. b.

Simplify the following expressions:

1. $\frac{2ab}{3cd} \times \frac{c^2d^3}{ab^2}$.
2. $\frac{12a^2bc}{8ab^3} \times \frac{24ab^2}{36bc^2}$.
3. $\frac{15xyz^3}{a^2bc} \times \frac{3a^3x}{5yz}$.
4. $\frac{7a^2b^3}{9ax^2y} \times \frac{18x^2c}{15ac^4}$.
5. $\frac{8m^2n^3}{5x^2yz} \times \frac{15xyz^3}{16mn^2}$.
6. $\frac{21k^2p^3}{13mn^2} \times \frac{39n^3m^2}{28p^2k^3}$.
7. $\frac{3a^2b}{4b^3c} \times \frac{2c^2}{8a^3} \div \frac{6ac}{16b^2x}$.
8. $\frac{2x^2y}{3yz} \times \frac{5z^2x}{7xy^2} \div \frac{21x^2y^3z^2}{40xy^2z}$.
9. $\frac{7m^2p}{17x^2y} \times \frac{51y^3z}{21p^2n} \div \frac{m^2x^2}{pyz}$.
10. $\frac{26xk^2p^3}{58mp^4} \times \frac{2xk^3}{13pkm} \div \frac{2x^2k^4}{87m^2p^2}$.
11. $\frac{15b^2}{40c} \times \frac{27c^2}{81d^3} \div \frac{abc}{14d^3}$.
12. $\frac{b^2}{3c} \times \frac{4c^2}{5d^3} \div \frac{16a^2b^2c^2}{15d^5}$.
13. $\frac{8ax^2}{7by} \times \frac{49cy^2}{64dx^3}$.
14. $\frac{15abc}{16xyz} \times \frac{128x^3y^2z^2}{100a^2bc}$.
15. $\frac{45a^2b^3c^4}{27x^4y^3z} \times \frac{243xy^2z^3}{180a^2bc^3}$.
16. $\frac{104xyzk^2p}{28xy^2kp^2} \times \frac{56y^3z^5p}{26y^2z^6k}$.
17. $\frac{m^2}{8n} \times \frac{36p^3q^2}{81mn} \div \frac{15mpx^5}{27n^2x^3y}$.
18. $\frac{a^3}{b^3} \times \frac{xy^2}{ab} \times \frac{pb^2}{ax} \div \frac{ap}{b^2}$.

Reduction to a Common Denominator.

98. In order to find the sum or difference of any fractions, we must, as in Arithmetic, first reduce them to a common denominator; and it is most convenient to take the lowest common multiple of the denominators of the given fractions.

Example. Express with lowest common denominator the fractions

$$\frac{a}{3xy},\ \frac{b}{6xyz},\ \frac{c}{2yz}.$$

The lowest common multiple of the denominators is $6xyz$. Multiplying the numerator of each fraction by the factor which is required to make its denominator $6xyz$, we have the equivalent fractions

$$\frac{2az}{6xyz},\ \frac{b}{6xyz},\ \frac{3cx}{6xyz}.$$

Note. The same result would clearly be obtained by dividing the lowest common denominator by each of the denominators in turn, and multiplying the corresponding numerators by the respective quotients.

EXAMPLES XII. c.

Express as equivalent fractions with common denominator:

1. $\frac{2x}{a},\ \frac{y}{2a}.$ **2.** $\frac{4x}{3y},\ \frac{y}{x^2}.$ **3.** $\frac{a}{2b},\ \frac{b}{c}.$ **4.** $\frac{a}{b},\ \frac{c}{d},\ 2.$

5. $\frac{2a}{b},\ \frac{b}{3c}.$ **6.** $\frac{m}{4n},\ \frac{p}{5n}.$ **7.** $\frac{k}{2x},\ \frac{p}{3x}.$ **8.** $\frac{m}{3x},\ \frac{n}{6x}.$

9. $\frac{a}{bc},\ \frac{b}{ca}.$ **10.** $\frac{a}{x},\ \frac{b}{x^2}.$ **11.** $\frac{2}{x},\ \frac{3}{y}.$ **12.** $\frac{x}{y},\ \frac{y}{x},\ 3x.$

13. $\frac{2x}{3y},\ \frac{3y}{2x}.$ **14.** $\frac{4a}{5b},\ \frac{3a}{10c}.$ **15.** $\frac{3a}{7b},\ \frac{5b}{21c}.$ **16.** $\frac{2}{a},\ \frac{b}{3},\ \frac{a}{9}.$

Addition and Subtraction of Fractions.

99. Rule. To add or subtract fractions: *express all the fractions with their lowest common denominator; form the algebraical sum of the numerators, and retain the common denominator.*

Example 1. Simplify $\frac{5x}{3}+\frac{3}{4}x-\frac{7x}{6}.$

The least common denominator is 12.

The expression $=\frac{20x+9x-14x}{12}=\frac{15x}{12}=\frac{5x}{4}.$

Example 2. Simplify $\dfrac{3ab}{5x} - \dfrac{ab}{2x} - \dfrac{1}{10} \cdot \dfrac{ab}{x}$.

The expression $= \dfrac{6ab - 5ab - ab}{10x} = \dfrac{0}{10x} = 0$.

Example 3. Simplify $\dfrac{2x}{a^2c^2} - \dfrac{y}{3ca^3}$.

The expression $= \dfrac{6ax - cy}{3a^3c^2}$, and admits of no further simplification.

Note. The beginner must be careful to distinguish between **erasing equal terms with different signs**, as in Example 2, and **cancelling equal factors** in the course of multiplication, or in reducing fractions to lowest terms. Moreover, in simplifying fractions he must remember that a factor can only be removed from numerator and denominator when it divides each *taken as a whole.*

Thus in $\dfrac{6ax - cy}{3a^3c^2}$, c cannot be cancelled because it only divides cy and not the *whole* numerator. Similarly a cannot be cancelled because it only divides $6ax$ and not the whole numerator. The fraction is therefore in its simplest form.

When no denominator is expressed the denominator 1 may be understood.

Example 4. $3x - \dfrac{a^2}{4y} = \dfrac{3x}{1} - \dfrac{a^2}{4y} = \dfrac{12xy - a^2}{4y}$,

If a fraction is not in its lowest terms it should be simplified before combining it with other fractions.

Example 5. $\dfrac{ax}{2} - \dfrac{x^2y}{3xy} = \dfrac{ax}{2} - \dfrac{x}{3} = \dfrac{3ax - 2x}{6}$.

EXAMPLES XII. d.

Simplify the following expressions :

1. $\dfrac{x}{2} + \dfrac{x}{3}$.	**2.** $\dfrac{y}{4} - \dfrac{y}{5}$.	**3.** $\dfrac{a}{3} - \dfrac{a}{4}$.	**4.** $\dfrac{2x}{3} - \dfrac{5}{x}$.
5. $\dfrac{x}{2} + \dfrac{y}{5}$.	**6.** $\dfrac{a}{4} - \dfrac{b}{6}$.	**7.** $\dfrac{m}{8} - \dfrac{n}{12}$.	**8.** $\dfrac{2m}{15} - \dfrac{n}{5}$.
9. $\dfrac{x}{7} - \dfrac{y}{21}$.	**10.** $\dfrac{a}{13} + \dfrac{b}{39}$.	**11.** $\dfrac{p}{16} - \dfrac{q}{48}$.	**12.** $\dfrac{5m}{12} - \dfrac{n}{36}$.
13. $\dfrac{2x}{3} + \dfrac{4x}{5}$.	**14.** $\dfrac{5x}{4} - \dfrac{4x}{5}$.	**15.** $\dfrac{5x}{6} - \dfrac{7x}{12}$.	**16.** $\dfrac{2a}{5} - \dfrac{4b}{15}$.

Simplify the following expressions :

17. $\frac{a}{2}-\frac{a}{3}+\frac{a}{5}$.
18. $\frac{x}{4}-\frac{x}{8}+\frac{x}{12}$.
19. $\frac{x}{3}+\frac{x}{6}-\frac{x}{9}$.
20. $\frac{2x}{3}-\frac{x}{6}+\frac{3x}{4}$.
21. $\frac{5x}{6}-\frac{x}{12}+\frac{x}{9}$.
22. $\frac{7x}{8}+\frac{x}{12}-\frac{x}{4}$.
23. $\frac{x}{a}-\frac{y}{b}$.
24. $\frac{3x^3}{ax^2}+\frac{2y}{3b}$.
25. $a+\frac{b}{c}$.
26. $x-\frac{y^2}{yz}$.
27. $\frac{a^3}{3a^2}-\frac{b^2}{a}$.
28. $a^2+\frac{b^3}{a}$.
29. $\frac{3x^2}{6x}-\frac{y^2}{x^2}$.
30. $p^3-\frac{k^5}{p^2}$.

MISCELLANEOUS EXAMPLES II.

(*Chiefly on* Chapters I.–VIII.)

[*The examples marked with an asterisk must be postponed by those who adopt the suggestions printed in italics on pages* 33 *and* 38.]

1. What expression must be added to $4x^3-3x^2+2$ to produce $4x^3+7x-6$?

2. If $A=6x-3y+2z$, $B=x+y+z$, and $C=10x+y-7z$, find the value of $A+4B-C$.

3. If $x=3$, $y=4$, $z=1$, find the value of

$$\sqrt{2xy+4xz}+\sqrt{9y}+\frac{2xyz}{3}.$$

4. Simplify by removing brackets

$$a^2+2d^2-(2e^2-b^2)-\{(d^2-c^2-e^2)+(d^2-e^2)\}.$$

***5.** Multiply x^3+x^2+3x+5 by x^2-x-2.

6. Solve the equations :

(1) $3-4x=36x-17$; (2) $5x-15=17x+21$.

***7.** Divide x^4-10x^2+9 by x^2-2x-3.

8. Simplify $7a-4b-\{5a-3[b-2(a-b)]\}$.

9. In an examination A has $x+y$ marks, B has $2x-3y$, and C has twice as many as A; how many marks have A, B, and C together?

10. Find the sum of $1-2x+x^2$, $3x-2x^2$, $5x^2-7x-2$, arranging the result in descending powers of x.

11. Write down the following products:

(1) $(x+17)(x-3)$; (2) $(3x-8)(8x+3)$.

12. Solve the equations:

(1) $7x-3-(7-5x)=3-3x-(5x+8)$;

(2) $(5x+1)(x-2)-(4x-3)(3x-1)=10-(7x+2)(x+1)$.

---

13. From the sum of $3ab$, $-5ab$, $2ab$, $7ab$, $-9ab$, subtract the sum of $-8ab$, $6ab$, $-9ab$, $10ab$.

14. When $a=4$, $b=3$, $c=2$, find the numerical value of

$$\frac{2a+b(2c-a)}{3b-\sqrt{2c^3}}.$$

15. From what expression must $11a^2-5ab-7bc$ be subtracted so as to give for remainder $7b(a+c)+5a^2$?

***16.** Multiply x^3+6x^2+8x-8 by x^2-2x+4.

17. Simplify

$$12a-[6a-2\{3a-4(b-a)\}-(9a+8b)].$$

18. Solve the equations:

(1) $3(2x-1)+2(3x-2)+3=4(x-5)$;

(2) $\frac{1}{3}(x+1)+\frac{1}{4}(x+3)=\frac{1}{5}(x+4)+16$.

Verify the solution in each case.

***19.** Divide $3p^5+16p^4-33p^3+14p^2$ by p^2+7p.

20. Add together

$a+2b-(2c+d)$, $3a-(b-2c)+2d$, and $2a-[b-(2c-3d)]$.

21. To what expression must $7x^3-6x^2-5x$ be added so as to make $9x^3-6x-7x^2$?

22. What value of x will make the product of $x+1$ and $2x+1$ less than the product of $x+3$ and $2x+3$ by 14?

23. When $a=2$, $b=3$, $c=1$, $q=4$, $r=6$, find the value of

$$5a^bc^r-3^a2^b+2^ra^5-c^bb^q.$$

24. Solve the equation:

$$x-\frac{x-13}{9}=\frac{6x+1}{5}+\frac{2}{3}\left(6-\frac{3x}{2}\right).$$

Shew also that $x=3$ does *not* satisfy the equation.

25. A horse can eat $3m+2n$ bushels of corn in a week; how many weeks will he be in eating $12m^2-7mn-10n^2$ bushels?

26. Subtract the sum of

$$2x^3-3x+4 \text{ and } -3x^2+2x-7$$

from
$$4x^3-3x^2+x-6-[2x^3-(x-6)].$$

27. Find the value of $a^3+b^3+c^3-3abc$, when $a=1$, $b=4$, $c=-5$.

28. Solve the equations:

$$(1)\quad \frac{2x}{15}+\frac{x-6}{12}=\frac{3}{10}\left(\frac{x}{2}-5\right);$$

$$(2)\quad \frac{2(x-1)}{5}+\frac{15}{2}\left(1-\frac{x}{3}\right)+\frac{19}{10}=\frac{9}{5}\left(\frac{x}{6}-\frac{1}{3}\right).$$

***29.** Divide $3y^6-37y^4+35y^3+7y^2+2$ by $y(y-1)(y+4)-2$.

30. Divide \$1120 between A and B so that for every half-dollar that A receives B may receive 20 cents.

---

31. Find the value of

$$(a+b)^2+(b+c)^2+(c+a)^2$$

when $a=-1$, $b=-2$, $c=3$.

32. Multiply $(2m^2+8)(m+2)$ by $3m-6$.

***33.** Divide the product of

$$x-2,\ x+3, \text{ and } 2x-7$$

by the sum of
$$3(x^2-2x-2) \text{ and } 5x-x^2-15.$$

34. A man walks at the rate of a miles an hour for p hours; he then rides for q hours at the rate of b miles an hour. How far has he travelled, and how long would it have taken to ride the same distance at c miles an hour?

Also work out the result supposing $p=7$, $q=3$, $a=4$, $b=9$, $c=11$.

35. Solve the equations:

$$(1)\quad \frac{3x}{2}-\frac{5}{7}=21x-\frac{1}{3}(2x+10\tfrac{3}{14});$$

$$(2)\quad 3x-4-\frac{4(7x-9)}{15}=\frac{4}{5}\left(6+\frac{x-1}{3}\right).$$

36. An estate was divided among three persons in such a way that the share of the first was three times that of the second, and the share of the second twice that of the third. The first received \$900 more than the third. How much did each receive?

CHAPTER XIII.

SIMULTANEOUS EQUATIONS.

[*In connection with this chapter the student may read* Chap. XXXV. Arts. 311–318.]

100. CONSIDER the equation $2x+5y=23$, which contains *two* unknown quantities.

From this we get $\qquad 5y=23-2x,$

that is,
$$y=\frac{23-2x}{5} \quad\text{.............................(1).}$$

From this it appears that for every value we choose to give to x there will be one corresponding value of y. Thus we shall be able to find as many pairs of values as we please which satisfy the given equation.

For instance, if $x=1$, then from (1) $y=\frac{21}{5}$.

Again, if $x=-2$, then $y=\frac{27}{5}$; and so on.

But if also we have a second equation of the same kind, such as
$$3x+4y=24,$$
we have from this
$$y=\frac{24-3x}{4} \quad\text{.............................(2).}$$

If now we seek values of x and y which satisfy *both* equations, the values of y in (1) and (2) must be identical.

Therefore
$$\frac{23-2x}{5}=\frac{24-3x}{4}.$$

Multiplying up,
$$92-8x=120-15x;$$
$$\therefore \quad 7x=28;$$
$$\therefore \quad x=4.$$

Substituting this value in the first equation, we have
$$8+5y=23;$$
$$\therefore \quad 5y=15;$$
$$\therefore \quad \left.\begin{aligned} y&=3,\\ x&=4.\end{aligned}\right\}$$
and

Thus, if both equations are to be satisfied by the *same* values of x and y, there is only one solution possible.

101. Definition. When two or more equations are satisfied by the same values of the unknown quantities they are called **simultaneous equations.**

We proceed to explain the different methods for solving simultaneous equations. In the present chapter we shall confine our attention to the simpler cases in which the unknown quantities are involved in the first degree.

102. In the example already worked we have used the method of solution which best illustrates the meaning of the term *simultaneous equation*; but in practice it will be found that this is rarely the readiest mode of solution. It must be borne in mind that since the two equations are simultaneously true, *any* equation formed by combining them will be satisfied by the values of x and y which satisfy the original equations. Our object will always be to obtain an equation which involves *one only* of the unknown quantities.

103. The process by which we get rid of either of the unknown quantities is called **elimination**, and it must be effected in different ways according to the nature of the equations proposed.

Example 1. Solve $3x+7y=27$ ....(1),

$5x+2y=16$ ....(2).

To eliminate x we multiply (1) by 5 and (2) by 3, so as to make the coefficients of x in both equations equal. This gives

$$15x+35y=135,$$
$$15x+\ \ 6y=48;$$

subtracting,
$$29y=87;$$
$$\therefore\ y=3.$$

To find x, substitute this value of y in *either* of the given equations.

Thus from (1)
$$3x+21=27;$$

and
$$\left.\begin{array}{r}\therefore\ x=2,\\ y=3.\end{array}\right\}$$

Note. When one of the unknowns has been found, it is immaterial which of the equations we use to complete the solution. Thus, in the present example, if we substitute 3 for y in (2), we have

$$5x+6=16;$$
$$\therefore\ x=2, \text{ as before.}$$

Example 2. Solve $7x+2y=47$ ....(1),

$5x-4y=1$ ....(2).

Here it will be more convenient to eliminate y.

Multiplying (1) by 2, $14x+4y=94$,

and from (2) $5x-4y=1$;

adding, $19x=95$;

$\therefore\ x=5$.

Substitute this value in (1),

$\therefore\ 35+2y=47$;

$$\left.\begin{aligned}\therefore\ y&=6,\\ \text{and}\qquad x&=5.\end{aligned}\right\}$$

Note. *Add* when the coefficients of one unknown are equal and *unlike* in sign; *subtract* when the coefficients are equal and *like* in sign.

Example 3. Solve $2x=5y+1$ ....(1),

$24-7x=3y$ ....(2).

Here we can eliminate x by substituting in (2) its value obtained from (1). Thus

$$24-\frac{7}{2}(5y+1)=3y;$$

$$\therefore\ 48-35y-7=6y;$$

$$\therefore\ 41=41y;$$

$$\left.\begin{aligned}\therefore\ y&=1,\\ \text{and from (1)}\qquad x&=3.\end{aligned}\right\}$$

104. Any one of the methods given above will be found sufficient; but there are certain arithmetical artifices which will frequently shorten the work.

Example 1. Solve $171x-213y=642$ ....(1),

$114x-326y=244$ ....(2).

Noticing that 171 and 114 contain a common factor 57, we shall make the coefficients of x in the two equations equal to the *least common multiple* of 171 and 114 if we multiply (1) by 2 and (2) by 3.

Thus $342x-426y=1284$,

$342x-978y=732$;

subtracting, $552y=552$;

that is, $y=1$,

and therefore from (1) $x=5$.

Example 2. Solve $127x + 59y = 1928$ ....(1),

$59x + 127y = 1792$ ....(2).

By addition, $186x + 186y = 3720$;

$\therefore\ x + y = 20$ ....(3).

Subtracting (2) from (1), $68x - 68y = 136$;

$\therefore\ x - y = 2$ ....(4).

Thus, by an easy combination of (1) and (2), the problem is reduced to the solution of the equations (3) and (4). From these we obtain by addition $2x = 22$, and by subtraction $2y = 18$.

Therefore $x = 11$, and $y = 9$.

EXAMPLES XIII. a.

[Art. 316 *may be read in connection with these Examples.*]

Solve the equations:

1. $3x + 4y = 10$, $4x + y = 9$.

2. $x + 2y = 13$, $3x + y = 14$,

3. $4x + 7y = 29$, $x + 3y = 11$.

4. $2x - y = 9$, $3x - 7y = 19$,

5. $5x + 6y = 17$, $6x + 5y = 16$.

6. $2x + y = 10$, $7x + 8y = 53$.

7. $8x - y = 34$, $x + 8y = 53$.

8. $15x + 7y = 29$, $9x + 15y = 39$.

9. $14x - 3y = 39$, $6x + 17y = 35$.

10. $28x - 23y = 33$, $63x - 25y = 101$.

11. $35x + 17y = 86$, $56x - 13y = 17$.

12. $15x + 77y = 92$, $55x - 33y = 22$.

13. $5x - 7y = 0$, $7x + 5y = 74$.

14. $21x - 50y = 60$, $28x - 27y = 199$.

15. $39x - 8y = 99$, $52x - 15y = 80$.

16. $5x = 7y - 21$, $21x - 9y = 75$.

17. $6y - 5x = 18$, $12x - 9y = 0$.

18. $8x = 5y$, $13x = 8y + 1$.

19. $3x = 7y$, $12y = 5x - 1$.

20. $19x + 17y = 0$, $2x - y = 53$.

21. $93x + 15y = 123$, $15x + 93y = 201$.

105. We add a few cases in which, before proceeding to solve, it will be necessary to simplify the equations.

Example 1. Solve $5(x + 2y) - (3x + 11y) = 14$ ....(1),

$7x - 9y - 3(x - 4y) = 38$ ....(2).

From (1) $5x + 10y - 3x - 11y = 14$;

$\therefore\ 2x - y = 14$ ....(3).

From (2) $7x - 9y - 3x + 12y = 38$;

$\therefore\ 4x + 3y = 38$ ....(4).

From (3) $6x - 3y = 42$.

By addition $10x = 80$; whence $x = 8$. From (3) we obtain $y = 2$.

Example 2. Solve $3x - \frac{y-5}{7} = \frac{4x-3}{2}$ ....(1),

$$\frac{3y+4}{5} - \frac{1}{3}(2x-5) = y \text{ ....(2).}$$

Clear of fractions. Thus

from (1) $42x - 2y + 10 = 28x - 21$;

$\therefore\ 14x - 2y = -31$ ....(3).

From (2) $9y + 12 - 10x + 25 = 15y$;

$\therefore\ 10x + 6y = 37$ ....(4).

Eliminating y from (3) and (4), we find that

$$x = -\frac{14}{13}.$$

Eliminating x from (3) and (4), we find that

$$y = \frac{207}{26}.$$

Note. Sometimes, as in the present instance, the value of the second unknown is more easily found by elimination than by substituting the value of the unknown already found.

EXAMPLES XIII. b.

Solve the equations :

1. $\frac{2x}{3} + y = 16,$ $\quad x + \frac{y}{4} = 14.$

2. $\frac{x}{5} + \frac{y}{2} = 5,$ $\quad x - y = 4.$

3. $\frac{5x}{6} - y = 3,$ $\quad x - \frac{5y}{6} = 8.$

4. $x - y = 5,$ $\quad \frac{x}{4} - \frac{y}{5} = 2.$

5. $\frac{x}{9} + \frac{y}{7} = 10,$ $\quad \frac{x}{3} + y = 50.$

6. $x = 3y,$ $\quad \frac{x}{3} + y = 34.$

7. $\frac{2}{5}x - \frac{1}{12}y = 3,$ $\quad 4x - y = 20.$

8. $\frac{1}{2}x - \frac{1}{5}y = 4,$ $\quad \frac{1}{7}x + \frac{1}{15}y = 3.$

9. $2x + y = 0,$ $\quad \frac{1}{2}y - 3x = 8.$

10. $\frac{x}{7} + \frac{y}{5} = 1\frac{3}{7},$ $\quad x + \frac{y}{3} = 4\frac{2}{3}.$

11. $3x - 7y = 0,$ $\quad \frac{2}{7}x + \frac{5}{3}y = 7.$

12. $\frac{x}{5} - \frac{y}{4} = 0,$ $\quad 3x + \frac{1}{2}y = 17.$

Solve the equations :

13. $\frac{x}{3}+\frac{y}{4}=3x-7y-37=0.$ **14.** $\frac{x+1}{10}=\frac{3y-5}{2}=\frac{x-y}{8}.$

15. $\frac{x+3}{5}=\frac{8-y}{4}=\frac{3(x+y)}{8}.$ **16.** $\frac{x}{13}-\frac{y}{7}=6x-10y-8=0.$

106. In order to solve simultaneous equations which contain two unknown quantities we have seen that we must have two equations. Similarly we find that in order to solve simultaneous equations which contain three unknown quantities we must have three equations.

Rule. *Eliminate one of the unknowns from any pair of the equations, and then eliminate the same unknown from another pair. Two equations involving two unknowns are thus obtained, which may be solved by the rules already given. The remaining unknown is then found by substituting in any one of the given equations.*

Example 1. Solve $6x+2y-5z=13$ .... (1),

$3x+3y-2z=13$ ....(2),

$7x+5y-3z=26$ ....(3).

Choose y as the unknown to be eliminated.

Multiply (1) by 3 and (2) by 2,

$$18x+6y-15z=39,$$
$$6x+6y-4z=26;$$

subtracting, $12x-11z=13$ ....(4).

Again, multiply (1) by 5 and (3) by 2,

$$30x+10y-25z=65,$$
$$14x+10y-6z=52;$$

subtracting, $16x-19z=13$ ....(5).

Multiply (4) by 4 and (5) by 3,

$$48x-44z=52,$$
$$48x-57z=39;$$

subtracting, $13z=13,$

$z=1,$

and from (4) $x=2,$

from (1) $y=3.$

Note. After a little practice the student will find that the solution may often be considerably shortened by a suitable combination of the proposed equations. Thus, in the present instance, by adding (1) and (2) and subtracting (3) we obtain $2x-4z=0$, or $x=2z$. Substituting in (1) and (2) we have two easy equations in y and z.

Some modification of the foregoing rule may often be used with advantage.

Example 2. Solve $\frac{x}{2}-1=\frac{y}{6}+1=\frac{z}{7}+2,$

$$\frac{y}{3}+\frac{z}{2}=13.$$

From the equation $\frac{x}{2}-1=\frac{y}{6}+1,$

we have $3x-y=12$ ....(1).

Also, from the equation $\frac{x}{2}-1=\frac{z}{7}+2,$

we have $7x-2z=42$ ....(2).

And, from the equation $\frac{y}{3}+\frac{z}{2}=13,$

we have $2y+3z=78$ ....(3).

Eliminating z from (2) and (3), we have

$$21x+4y=282\,;$$

and from (1) $12x-4y=48$;

whence $x=10,\ y=18.$

Also by substitution in (2) we obtain $z=14$.

Example 3. Consider the equations

$$5x-3y-\ \ z=6 \text{ ....(1)},$$
$$13x-7y+3z=14 \text{ ....(2)},$$
$$7x-4y=8 \text{ ....(3)}.$$

Multiplying (1) by 3 and adding to (2), we have

$$28x-16y=32,$$

or $7x-\ 4y=8.$

Thus the combination of equations (1) and (2) leads us to an equation which is identical with (3), and so to find x and y we have but a single equation $7x-4y=8$, the solution of which is indeterminate. [Art. 100.]

In this and similar cases the anomaly arises from the fact that the equations are not *independent* ; in other words, one equation is deducible from the others, and therefore contains no new relation between the unknown quantities which is not already implied in the other equations.

EXAMPLES XIII. c.

Solve the equations :

1. $$\begin{aligned} x+2y+2z&=11,\\ 2x+y+z&=7,\\ 3x+4y+z&=14. \end{aligned}$$

2. $$\begin{aligned} x+3y+4z&=14,\\ x+2y+z&=7,\\ 2x+y+2z&=2. \end{aligned}$$

3. $$\begin{aligned} x+4y+3z&=17,\\ 3x+3y+z&=16,\\ 2x+2y+z&=11. \end{aligned}$$

4. $$\begin{aligned} 3x-2y+z&=2,\\ 2x+3y-z&=5,\\ x+y+z&=6. \end{aligned}$$

5. $$\begin{aligned} 2x+y+z&=16,\\ x+2y+z&=9,\\ x+y+2z&=3. \end{aligned}$$

6. $$\begin{aligned} x-2y+3z&=2,\\ 2x-3y+z&=1,\\ 3x-y+2z&=9. \end{aligned}$$

7. $$\begin{aligned} 3x+2y-z&=20,\\ 2x+3y+6z&=70,\\ x-y+6z&=41. \end{aligned}$$

8. $$\begin{aligned} 2x+3y+4z&=20,\\ 3x+4y+5z&=26,\\ 3x+5y+6z&=31. \end{aligned}$$

9. $3x-4y=6z-16,\ \ 4x-y-z=5,\ \ x=3y+2(z-1).$

10. $5x+2y=14,\ \ y-6z=-15,\ \ x+2y+z=0.$

11. $x-\dfrac{y}{5}=6,\ \ y-\dfrac{z}{7}=8,\ \ z-\dfrac{x}{2}=10.$

12. $\dfrac{y+z}{4}=\dfrac{z+x}{3}=\dfrac{x+y}{2},\ \ x+y+z=27.$

13. $\dfrac{y-z}{3}=\dfrac{y-x}{2}=5z-4x,\ \ y+z=2x+1.$

14. $2x+3y=5,\ \ 2z-y=1,\ \ 7x-9z=3.$

15. $$\begin{aligned} \tfrac{1}{2}(x+z-5)&=y-z\\ &=2x-11\\ &=9-(x+2z). \end{aligned}$$

16. $$\begin{aligned} x+20&=\frac{3y}{2}+10\\ &=2z+5\\ &=110-(y+z). \end{aligned}$$

***107.** Definition. If the product of two quantities be equal to unity, each is said to be the **reciprocal** of the other. Thus if $ab=1$, a and b are **reciprocals**. They are so called because $a=\dfrac{1}{b}$, and $b=\dfrac{1}{a}$; and consequently a is related to b exactly as b is related to a.

The reciprocals of x and y are $\dfrac{1}{x}$ and $\dfrac{1}{y}$ respectively, and in solving the following equations we consider $\dfrac{1}{x}$ and $\dfrac{1}{y}$ as the unknown quantities.

Example 1. Solve $$\frac{8}{x}-\frac{9}{y}=1 \quad\dots\dots(1),$$
$$\frac{10}{x}+\frac{6}{y}=7 \quad\dots\dots(2).$$

Multiply (1) by 2 and (2) by 3; thus
$$\frac{16}{x}-\frac{18}{y}=2,$$
$$\frac{30}{x}+\frac{18}{y}=21;$$

adding, $$\frac{46}{x}=23;$$

multiplying up, $$46=23x,$$
$$x=2;$$

and by substituting in (1), $$y=3.$$

Example 2. Solve $$\frac{1}{2x}+\frac{1}{4y}-\frac{1}{3z}=\frac{1}{4} \quad\dots\dots(1),$$
$$\frac{1}{x}=\frac{1}{3y} \quad\dots\dots(2),$$
$$\frac{1}{x}-\frac{1}{5y}+\frac{4}{z}=2\tfrac{2}{15} \quad\dots\dots(3),$$

clearing of fractional coefficients, we obtain

from (1) $$\frac{6}{x}+\frac{3}{y}-\frac{4}{z}=3 \quad\dots\dots(4),$$

from (2) $$\frac{3}{x}-\frac{1}{y}\qquad=0 \quad\dots\dots(5),$$

from (3) $$\frac{15}{x}-\frac{3}{y}+\frac{60}{z}=32 \quad\dots\dots(6).$$

Multiply (4) by 15 and add the result to (6); we have
$$\frac{105}{x}+\frac{42}{y}=77;$$

dividing by 7, $$\frac{15}{x}+\frac{6}{y}=11 \quad\dots\dots(7);$$

from (5) $$\frac{18}{x}-\frac{6}{y}=0;$$
$$\therefore\ \frac{33}{x}=11;$$
$$\therefore\ \left.\begin{aligned} x&=3,\\ y&=1,\\ z&=2.\end{aligned}\right\}$$

from (5)

from (4)

*EXAMPLES. XIII. d.

Solve the equations :

1. $\dfrac{5}{x}+\dfrac{6}{y}=3,$
$\dfrac{15}{x}+\dfrac{3}{y}=4.$

2. $\dfrac{6}{x}-\dfrac{7}{y}=2,$
$\dfrac{2}{x}+\dfrac{14}{y}=3.$

3. $\dfrac{12}{x}-\dfrac{4}{y}=2,$
$\dfrac{3}{x}-\dfrac{2}{y}=0.$

4. $\dfrac{5}{x}+\dfrac{16}{y}=79,$
$\dfrac{16}{x}-\dfrac{1}{y}=44.$

5. $\dfrac{21}{x}+\dfrac{12}{y}=5,$
$\dfrac{1}{y}-\dfrac{1}{x}=\dfrac{1}{42}.$

6. $\dfrac{5}{x}+\dfrac{3}{y}=30,$
$\dfrac{9}{x}-\dfrac{5}{y}=2.$

7. $\dfrac{8}{x}-\dfrac{9}{y}=7,$
$6\left(\dfrac{1}{x}+\dfrac{1}{y}\right)=1.$

8. $\dfrac{25}{x}+\dfrac{24}{y}=1,$
$20\left(\dfrac{2}{x}+\dfrac{3}{y}\right)=7.$

9. $\dfrac{4}{x}+\dfrac{27}{y}=42,$
$\dfrac{14}{x}-\dfrac{15}{y}=1.$

10. $\dfrac{3}{x}+\dfrac{5}{y}=\dfrac{8}{15},$
$9y-22x=\dfrac{3xy}{25}.$

11. $\dfrac{1}{4x}+\dfrac{1}{3y}=2,$
$\dfrac{1}{y}-\dfrac{1}{2x}=1.$

12. $2y-x=4xy,$
$\dfrac{4}{y}-\dfrac{3}{x}=9.$

13. $\dfrac{1}{x}-\dfrac{2}{y}+4=0,$
$\dfrac{1}{y}-\dfrac{1}{z}+1=0,$
$\dfrac{2}{z}+\dfrac{3}{x}=14.$

14. $\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}=36,$
$\dfrac{1}{x}+\dfrac{3}{y}-\dfrac{1}{z}=28,$
$\dfrac{1}{x}+\dfrac{1}{3y}+\dfrac{1}{2z}=20.$

15. $\dfrac{9}{x}-\dfrac{2}{y}=\dfrac{5}{z}-\dfrac{3}{x}=\dfrac{7}{y}+\dfrac{15}{2z}=4.$

CHAPTER XIV.

PROBLEMS LEADING TO SIMULTANEOUS EQUATIONS.

108. IN the Examples discussed in the last chapter we have seen that it is essential to have as many equations as there are unknown quantities to determine. Consequently in the solution of problems which give rise to simultaneous equations, it will always be necessary that the statement of the question should contain as many independent conditions as there are quantities to be determined.

Example 1. Find two numbers whose difference is 11, and one-fifth of whose sum is 9.

Let x be the greater number, y the less;

Then $$x-y=11 \quad\text{.................(1)},$$

Also $$\frac{x+y}{5}=9,$$

or $$x+y=45 \quad\text{.................(2)}.$$

By addition $2x=56$; and by subtraction $2y=34$.
The numbers are therefore 28 and 17.

Example 2. If 15 lbs. of tea and 10 lbs. of coffee together cost \$15.50, and 25 lbs. of tea and 13 lbs. of coffee together cost \$24.55, find the price of each per pound.

Suppose a pound of tea to cost x cents and a pound of coffee to cost y cents.

Then from the question, we have

$$15x+10y=1550 \quad\text{.................(1)},$$
$$25x+13y=2455 \quad\text{.................(2)}.$$

Multiplying (1) by 5 and (2) by 3, we have

$$75x+50y=7750,$$
$$75x+39y=7365.$$

Subtracting, $$11y=385,$$
$$y=35.$$

And from (1) $$15x+350=1550.$$

Whence $$15x=1200;$$
$$\therefore\ x=80.$$

Therefore the cost of a pound of tea is 80 cents, and the cost of a pound of coffee is 35 cents.

Example 3. A person spent \$6.80 in buying oranges at the rate of 3 for 10 cents, and apples at 15 cents a dozen; if he had bought five times as many oranges and a quarter of the number of apples, he would have spent \$25.45. How many of each did he buy?

Let x represent the number of oranges and y the number of apples.

$$x \text{ oranges cost } \frac{10x}{3} \text{ cents,}$$

$$y \text{ apples cost } \frac{15y}{12} \text{ cents;}$$

$$\therefore \quad \frac{10x}{3}+\frac{15y}{12}=680 \quad \text{.........(1)}$$

Again, $5x$ oranges cost $5x \times \frac{10}{3}$, or $\frac{50x}{3}$ cents, and $\frac{y}{4}$ apples cost $\frac{y}{4} \times \frac{15}{12}$, or $\frac{15y}{48}$ cents;

$$\therefore \quad \frac{50x}{3}+\frac{15y}{48}=2545 \quad \text{.........(2).}$$

Multiply (1) by 5 and subtract (2) from the result;

then $$\left(\frac{75}{12}-\frac{15}{48}\right)y=855;$$

or $$\frac{285y}{48}=855;$$

$$\therefore \quad y=144;$$

and from (1) $$x=150.$$

Thus there were 150 oranges and 144 apples.

Example 4. If the numerator of a fraction is increased by 2 and the denominator by 1, it becomes equal to $\frac{5}{8}$; and, if the numerator and denominator are each diminished by 1, it becomes equal to $\frac{1}{2}$: find the fraction.

Let x be the numerator of the fraction, y the denominator; then the fraction is $\frac{x}{y}$.

From the first supposition,

$$\frac{x+2}{y+1}=\frac{5}{8} \quad \text{.........(1),}$$

from the second, $$\frac{x-1}{y-1}=\frac{1}{2} \quad \text{.........(2).}$$

These equations give $x=8$, $y=15$.

Thus the fraction is $\frac{8}{15}$.

Example 5. The middle digit of a number between 100 and 1000 is zero, and the sum of the other digits is 11. If the digits be reversed, the number so formed exceeds the original number by 495; find it.

Let x be the digit in the units' place,

y .... hundreds' place;

then, since the digit in the tens' place is 0, the number will be represented by $100y+x$. [Art. 81, Ex. 4.]

And if the digits are reversed the number so formed will be represented by $100x+y$.

$$\therefore\ 100x+y-(100y+x)=495,$$

or

$$100x+y-100y-x=495;$$

$$\therefore\ 99x-99y=495,$$

that is,

$$x-y=5 \quad\text{.........................(1)},$$

Again, since the sum of the digits is 11, and the middle one is 0, we have

$$x+y=11 \quad\text{.........................(2)}.$$

From (1) and (2) we find $x=8$, $y=3$.

Hence the number is 308.

EXAMPLES XIV.

1. Find two numbers whose sum is 34, and whose difference is 10.

2. The sum of two numbers is 73, and their difference is 37; find the numbers.

3. One third of the sum of two numbers is 14, and one half of their difference is 4; find the numbers.

4. One nineteenth of the sum of two numbers is 4, and their difference is 30; find the numbers.

5. Half the sum of two numbers is 20, and three times their difference is 18; find the numbers.

6. Six pounds of tea and eleven pounds of sugar cost \$5.65, and eleven pounds of tea and six pounds of sugar cost \$9.65. Find the cost of tea and sugar per pound.

7. Six horses and seven cows can be bought for \$250, and thirteen cows and eleven horses can be bought for \$461. What is the value of each animal?

8. A, B, C, D have \$290 between them; A has twice as much as C, and B has three times as much as D; also C and D together have \$50 less than A. Find how much each has.

9. A, B, C, D have \$270 between them; A has three times as much as C, and B five times as much as D; also A and B together have \$50 less than eight times what C has. Find how much each has.

10. Four times B's age exceeds A's age by twenty years, and one third of A's age is less than B's age by two years: find their ages.

11. One eleventh of A's age is greater by two years than one seventh of B's, and twice B's age is equal to what A's age was thirteen years ago: find their ages.

12. In eight hours A walks twelve miles more than B does in seven hours; and in thirteen hours B walks seven miles more than A does in nine hours. How many miles does each walk per hour?

13. In eleven hours C walks $12\frac{1}{2}$ miles less than D does in twelve hours; and in five hours D walks $3\frac{1}{4}$ miles less than C does in seven hours. How many miles does each walk per hour?

14. Find a fraction such that if 1 be added to its denominator it reduces to $\frac{1}{2}$, and reduces to $\frac{3}{5}$ on adding 2 to its numerator.

15. Find a fraction which becomes $\frac{1}{2}$ on subtracting 1 from the numerator and adding 2 to the denominator, and reduces to $\frac{1}{3}$ on subtracting 7 from the numerator and 2 from the denominator.

16. If 1 be added to the numerator of a fraction it reduces to $\frac{1}{5}$; if 1 be taken from the denominator it reduces to $\frac{1}{7}$: required the fraction.

17. If $\frac{2}{3}$ be added to the numerator of a certain fraction the fraction will be increased by $\frac{1}{21}$, and if $\frac{1}{2}$ be taken from its denominator the fraction becomes $\frac{2}{9}$: find it.

18. The sum of a number of two digits and of the number formed by reversing the digits is 110, and the difference of the digits is 6: find the numbers.

19. The sum of the digits of a number is 13, and the difference between the number and that formed by reversing the digits is 27: find the numbers.

20. A certain number of two digits is three times the sum of its digits, and if 45 be added to it the digits will be reversed: find the number.

21. A certain number between 10 and 100 is eight times the sum of its digits, and if 45 be subtracted from it the digits will be reversed: find the number.

22. A man has a number of dollar bills and ten-cent pieces, and he observes that if the dollars were turned into ten-cent pieces and the ten-cent pieces into dollars he would gain \$2.70; but if the dollars were turned into half-dollars and the ten-cent pieces into quarters he would lose \$1.30. How many of each had he?

23. In a bag containing black and white balls, half the number of white is equal to a third of the number of black; and twice the whole number of balls exceeds three times the number of black balls by four. How many balls did the bag contain?

24. A number consists of three digits, the right-hand one being zero. If the left-hand and middle digits be interchanged the number is diminished by 180; if the left-hand digit be halved and the middle and right-hand digits be interchanged the number is diminished by 454. Find the number.

25. The wages of 10 men and 8 boys amount to \$22.30; if 4 men together receive \$3.40 more than 6 boys, what are the wages of each man and boy?

26. A grocer wishes to mix sugar at 8 cents a pound with another sort at 5 cents a pound to make 60 pounds to be sold at 6 cents a pound. What quantity of each must he take?

27. A traveller walks a certain distance; had he gone half a mile an hour faster, he would have walked it in four-fifths of the time: had he gone half a mile an hour slower, he would have been $2\frac{1}{2}$ hours longer on the road. Find the distance.

28. A man walks 35 miles partly at the rate of 4 miles an hour, and partly at 5; if he had walked at 5 miles an hour when he walked at 4, and vice versâ, he would have covered two miles more in the same time. Find the time he was walking.

29. Two persons, 27 miles apart, setting out at the same time are together in 9 hours if they walk in the same direction, but in 3 hours if they walk in opposite directions: find their rates of walking.

30. When a certain number of two digits is doubled, and increased by 10, the result is the same as if the number had been reversed, and doubled, and then diminished by 8; also the number itself exceeds 3 times the sum of its digits by 18: find the number.

31. If I lend a sum of money at 6 per cent., the interest for a certain time exceeds the loan by \$100; but if I lend it at 3 per cent., for a fourth of the time, the loan exceeds its interest by \$425. How much do I lend?

32. A takes 3 hours longer than B to walk 30 miles; but if he doubles his pace he takes 2 hours less time than B: find their rates of walking.

CHAPTER XV.

INVOLUTION.

[Arts. 41–45 *should be studied here by those who have adopted the postponement suggested on page* 33.]

109. DEFINITION. **Involution** is the general name for multiplying an expression by itself so as to find its second, third, fourth, or any other power.

Involution may always be effected by actual multiplication. Here, however, we shall give some rules for writing down at once

(1) any power of a simple expression;

(2) the square and cube of any binomial;

(3) the square of any multinomial.

110. It is evident from the Rule of Signs that

(1) no *even* power of *any* quantity can be *negative*;

(2) any *odd* power of a quantity will have *the same sign* as the quantity itself.

Note. It is especially worthy of notice that the *square* of every expression, whether positive or negative, is *positive*.

111. From definition we have, by the rules of multiplication,

$$(a^2)^3 = a^2 \,.\, a^2 \,.\, a^2 = a^{2+2+2} = a^6.$$

$$(-x^3)^2 = (-x^3)(-x^3) = x^{3+3} = x^6.$$

$$(-a^5)^3 = (-a^5)(-a^5)(-a^5) = -a^{5+5+5} = -a^{15}.$$

$$(-3a^3)^4 = (-3)^4(a^3)^4 = 81a^{12}.$$

Hence we obtain a rule for raising a simple expression to any proposed power.

Rule. (1) *Raise the coefficient to the required power by Arithmetic, and prefix the proper sign found by the Rule of Signs.*

(2) *Multiply the index of every factor of the expression by the exponent of the power required.*

Examples.

$$(-2x^2)^5 = -32x^{10}.$$
$$(-3ab^3)^6 = 729a^6b^{18}.$$
$$\left(\frac{2ab^3}{3x^2y}\right)^4 = \frac{16a^4b^{12}}{81x^8y^4}.$$

It will be seen that in the last case the numerator and the denominator are operated upon separately.

EXAMPLES XV. a.

Write down the square of each of the following expressions:

1. $3ab^3$. 2. a^3c. 3. $7ab^2$. 4. $11b^2c^3$.
5. $4a^4b^5x^2$. 6. $5x^2y^5$. 7. $-2abc^2$. 8. $-3cx^3$.
9. $4xyz^3$. 10. $-\frac{2}{3}a^2b^3$. 11. $\frac{2x^2}{3y^3}$. 12. $-\frac{4}{3x^2y}$.
13. $-\frac{7ab}{3}$. 14. $\frac{3a^2b^3}{4c^5x^4}$. 15. $-\frac{1}{2xy}$. 16. $-2xy^2$.
17. $\frac{5ab^3}{2xy}$. 18. $13c^5x^3$. 19. $-\frac{1}{4a^4}$. 20. $-\frac{3a^5}{5x^3}$.

Write down the cube of each of the following expressions:

21. $2ab^2$. 22. $3x^3$. 23. $4x^4$. 24. $-3a^3b$.
25. $-5ab^2$. 26. $-b^3c^2x$. 27. $-6a^6$. 28. $-2a^7c^2$.
29. $\frac{1}{3y^2}$. 30. $-\frac{3x^5}{5a^3}$. 31. $7x^3y^4$. 32. $-\frac{2}{3}a^5$.

Write down the value of each of the following expressions:

33. $(3a^2b^3)^4$. 34. $(-a^2x)^6$. 35. $(-2x^3y)^5$. 36. $\left(\frac{1}{2a^2}\right)^7$.
37. $\left(\frac{3x^4}{2y^3}\right)^5$. 38. $\left(\frac{2x^3}{3y}\right)^8$. 39. $\left(-\frac{x^3}{3}\right)^7$. 40. $\left(-\frac{2x^5}{3a^4}\right)^6$.

112. By multiplication we have

$$(a+b)^2 = (a+b)(a+b)$$
$$= a^2 + 2ab + b^2 \quad\text{.............}(1),$$
$$(a-b)^2 = (a-b)(a-b)$$
$$= a^2 - 2ab + b^2 \quad\text{.............}(2).$$

These results are embodied in the following rules:

Rule 1. *The square of the sum of two quantities is equal to the sum of their squares increased by twice their product.*

Rule 2. *The square of the difference of two quantities is equal to the sum of their squares diminished by twice their product.*

Example 1. $(x+2y)^2 = x^2 + 2 \cdot x \cdot 2y + (2y)^2$
$= x^2 + 4xy + 4y^2.$

Example 2. $(2a^3 - 3b^2)^2 = (2a^3)^2 - 2 \cdot 2a^3 \cdot 3b^2 + (3b^2)^2$
$= 4a^6 - 12a^3b^2 + 9b^4.$

113. These rules may sometimes be conveniently applied to find the squares of numerical quantities.

Example 1. The square of $1012 = (1000+12)^2$
$= (1000)^2 + 2 \cdot 1000 \cdot 12 + (12)^2$
$= 1000000 + 24000 + 144$
$= 1024144.$

Example 2. The square of $98 = (100-2)^2$
$= (100)^2 - 2 \cdot 100 \cdot 2 + (2)^2$
$= 10000 - 400 + 4$
$= 9604.$

The work is considerably shortened by the omission of the first two steps.

114. We may now extend the rules of Art 112 thus:

$$\begin{aligned}(a+b+c)^2 &= \{(a+b)+c\}^2\\ &= (a+b)^2 + 2(a+b)c + c^2 \qquad \text{[Art. 112. Rule 1.]}\\ &= a^2+b^2+c^2+2ab+2ac+2bc.\end{aligned}$$

In the same way we may prove

$$(a-b+c)^2 = a^2+b^2+c^2-2ab+2ac-2bc$$
$$(a+b+c+d)^2 = a^2+b^2+c^2+d^2+2ab+2ac+2ad+2bc+2bd+2cd.$$

In each of these instances we observe that the square consists of

(1) the sum of the squares of the several terms of the given expression;

(2) twice the sum of the products two and two of the several terms, taken with their proper signs; that is, in each product the sign is + or − according as the quantities composing it have like or unlike signs.

Note. The *square terms* are always positive.

The same laws hold whatever be the number of terms in the expression to be squared.

Rule. *To find the square of any multinominal: to the sum of the squares of the several terms add twice the product (with the proper sign) of each term into each of the terms that follow it.*

Ex. 1. $(x-2y-3z)^2 = x^2+4y^2+9z^2-2 \,.\, x \,.\, 2y - 2 \,.\, x \,.\, 3z + 2 \,.\, 2y \,.\, 3z$
$$= x^2+4y^2+9z^2-4xy-6xz+12yz.$$

Ex. 2. $(1+2x-3x^2)^2 = 1+4x^2+9x^4+2 \,.\, 1 \,.\, 2x - 2 \,.\, 1 \,.\, 3x^2 - 2 \,.\, 2x \,.\, 3x^2$
$$= 1+4x^2+9x^4+4x-6x^2-12x^3$$
$$= 1+4x-2x^2-12x^3+9x^4,$$
by collecting like terms and rearranging.

EXAMPLES XV. b.

Write down the square of each of the following expressions:

1. $a+3b$. 2. $a-3b$. 3. $x-5y$. 4. $2x+3y$.
5. $3x-y$. 6. $3x+5y$. 7. $9x-2y$. 8. $5ab-c$.
9. $pq-r$. 10. $x-abc$. 11. $ax+2by$. 12. x^2-1.
13. $a-b-c$. 14. $a+b-c$. 15. $a+2b+c$.
16. $2a-3b+4c$. 17. $x^2-y^2-z^2$. 18. $xy+yz+zx$.
19. $3p-2q+4r$. 20. x^2-x+1. 21. $2x^2+3x-1$.
22. $x-y+a-b$. 23. $2x+3y+a-2b$. 24. $m-n-p-q$.
25. $\frac{1}{2}a-2b+\frac{c}{4}$. 26. $\frac{a}{3}-3b-\frac{3}{2}$. 27. $\frac{2}{3}x^2-x+\frac{3}{2}$.

115. By actual multiplication we have
$$(a+b)^3=(a+b)(a+b)(a+b)$$
$$=a^3+3a^2b+3ab^2+b^3.$$
Also $$(a-b)^3=a^3-3a^2b+3ab^2-b^3.$$

By observing the law of formation of the terms in these results we can write down the cube of any binomial.

Example 1. $(2x+y)^3=(2x)^3+3(2x)^2y+3(2x)y^2+y^3$
$$=8x^3+12x^2y+6xy^2+y^3.$$

Example 2. $(3x-2a^2)^3=(3x)^3-3(3x)^2(2a^2)+3(3x)(2a^2)^2-(2a^2)^3$
$$=27x^3-54x^2a^2+36xa^4-8a^6.$$

EXAMPLES XV c.

Write down the cube of each of the following expressions:

1. $x+a$. 2. $x-a$. 3. $x-2y$. 4. $2x+y$.
5. $3x-5y$. 6. $ab+c$. 7. $2ab-3c$. 8. $5a-bc$.
9. x^2+4y^2. 10. $4x^2-5y^2$. 11. $2a^3-3b^2$. 12. $5x^5-4y^4$.
13. $a-\frac{2b}{3}$. 14. $\frac{a}{3}+2$. 15. $\frac{x^2}{3}-3x$. 16. $\frac{a}{6}+2x$.

CHAPTER XVI.

EVOLUTION.

[Arts. 51–54 *should be studied here by those who have adopted the postponement suggested on page* 38.]

116. DEFINITION. The **root** of any proposed expression is that quantity which being multiplied by itself the requisite number of times produces the given expression.

The operation of finding the root is called **Evolution**: it is the reverse of Involution.

117. By the Rule of Signs we see that

(1) any *even* root of a *positive* quantity may be either *positive* or *negative*;

(2) *no negative* quantity can have an *even* root;

(3) every *odd* root of a quantity has the same sign as the quantity itself.

Note. It is especially worthy of remark that every positive quantity has two square roots equal in magnitude, but opposite in sign.

Example. $\sqrt{9a^2x^6} = \pm 3ax^3$.

In the present chapter, however, we shall confine our attention to the positive root.

Examples. $\sqrt{a^6b^4} = a^3b^2$, because $(a^3b^2)^2 = a^6b^4$.

$\sqrt[3]{-x^9} = -x^3$, because $(-x^3)^3 = -x^9$.

$\sqrt[5]{c^{20}} = c^4$, because $(c^4)^5 = c^{20}$.

$\sqrt[4]{81x^{12}} = 3x^3$, because $(3x^3)^4 = 81x^{12}$.

118. From the foregoing examples we may deduce a general rule for extracting any proposed root of a simple expression:

Rule. (1) *Find the root of the coefficient by Arithmetic, and prefix the proper sign.*

(2) *Divide the exponent of every factor of the expression by the index of the proposed root.*

Examples.

$$\sqrt[3]{-64x^6} = -4x^2.$$

$$\sqrt[4]{16a^8} = 2a^2.$$

$$\sqrt{\frac{81x^{10}}{25c^4}} = \frac{9x^5}{5c^2}.$$

EXAMPLES XVI. a.

Write down the square root of the following expressions :

1. $4a^2b^4$.
2. $9x^6y^2$.
3. $25x^4y^6$.
4. $16a^4b^2c^6$.
5. $81a^6b^8$.
6. $100x^8$.
7. $a^{20}b^{16}c^4$.
8. $a^8b^2c^{12}$.
9. $64x^6y^{18}$.
10. $\dfrac{36}{a^{36}}$.
11. $\dfrac{a^{16}b^8}{16}$.
12. $\dfrac{289y^4}{25}$.
13. $\dfrac{324x^{12}}{169y^6}$.
14. $\dfrac{81a^{18}}{36b^{12}}$.
15. $\dfrac{256x^2y^4}{289p^{14}}$.
16. $\dfrac{400a^{40}b^{20}}{81x^{10}y^{18}}$.

Write down the cube root of the following expressions :

17. $27a^6b^3c^3$.
18. $-8a^{12}b^9$.
19. $64x^6y^3z^{12}$.
20. $-343a^{12}b^{18}$.
21. $-\dfrac{x^{12}y^9}{125}$.
22. $\dfrac{8x^9}{729y^{15}}$.
23. $\dfrac{125a^3b^6}{216x^6y^9}$.
24. $-\dfrac{27x^{27}}{64y^{63}}$.

Write down the value of each of the following expressions :

25. $\sqrt[4]{(a^8x^{12})}$.
26. $\sqrt[7]{(x^{14}y^{21})}$.
27. $\sqrt[5]{(32x^5y^{10})}$.
28. $\sqrt[6]{729a^{18}b^6)}$.
29. $\sqrt[8]{(256a^8x^{64})}$.
30. $\sqrt[5]{(-x^{10}y^{15})}$.
31. $\sqrt[7]{\dfrac{128}{a^{63}b^{56}}}$.
32. $\sqrt[10]{\dfrac{a^{30}x^{50}}{b^{100}}}$.
33. $\sqrt[9]{\dfrac{a^{18}}{b^{27}c^{36}}}$.

118A. By Art. 112, we can write down the square of any binomial.

Thus $\qquad (2x+3y)^2=(2x)^2+2 \,.\, 2x \,.\, 3y+(3y)^2$.

Conversely, by observing the form of the terms of an expression, its square root may often be written down at once.

Example 1. Find the square root of $25x^2-40xy+16y^2$.

$$\begin{aligned}\text{The expression} &= (5x)^2-2 \,.\, 20xy+(4y)^2\\ &=(5x)^2-2(5x)(4y)+(4y)^2\\ &=(5x-4y)^2.\end{aligned}$$

Thus the required square root is $5x-4y$.

Example 2. Find the square root of $\dfrac{64a^2}{9b^2}+4+\dfrac{32a}{3b}$.

$$\begin{aligned}\text{The expression} &= \left(\frac{8a}{3b}\right)^2+(2)^2+2\left(\frac{16a}{3b}\right)\\ &=\left(\frac{8a}{3b}\right)^2+2\left(\frac{8a}{3b}\right)(2)+(2)^2\\ &=\left(\frac{8a}{3b}+2\right)^2.\end{aligned}$$

Thus the required square root is $\dfrac{8a}{3b}+2$.

Example 3. Find the square root of $4a^2+b^2+c^2+4ab-4ac-2bc$.

Arrange the terms in descending powers of a, and the other letters alphabetically; then

$$\begin{aligned}\text{the expression} &= 4a^2+4ab-4ac+b^2-2bc+c^2\\ &= 4a^2+4a(b-c)+(b-c)^2\\ &= (2a)^2+2\,.\,2a(b-c)+(b-c)^2\\ &= \{2a+(b-c)\}^2;\end{aligned}$$

whence the required square root is $2a+b-c$,

Or we might proceed as follows:

$$\text{the expression} = (2a)^2+b^2+c^2+2\,.\,(2a)b-2\,.\,(2a)c-2\,.\,b\,.\,c,$$

which is evidently the square root of $2a+b-c$. [Art. 114.]

119. When the square root cannot be easily determined by inspection we must have recourse to the rule we are about to explain, which is quite general, and applicable to all cases. *But the student is advised to use methods of inspection wherever possible, in preference to rules.*

Since the square of $a+b$ is $a^2+2ab+b^2$, we have to discover a process by which a and b, the terms of the root, can be found when $a^2+2ab+b^2$ is given.

Now $$a^2+2ab+b^2=a^2+b(2a+b),$$

so that the expression is made up of

(1) the *square* of the *first* term of the root, together with

(2) the *product* of the *second* term of the root into an expression consisting of *this second term added to twice the first term of the root.*

By reversing the process we arrive at the following method of working:

$$\begin{array}{r|l} & a^2+2ab+b^2\,(\,a+b \\ & a^2 \\ \hline 2a+b & 2ab+b^2 \\ & 2ab+b^2 \\ \hline \end{array}$$

Explanation. (1) The terms are first arranged according to the powers of one letter a.

(2) The square root of a^2 is written down as *the first term of the root*, and its square subtracted from the given expression.

(3) The first term of the remainder is *divided by twice the first term of the root* to obtain the second term of the root, that is, b.

(4) *The second term of the root is added to twice the term already found* to form the complete divisor $2a+b$.

Example 1. Find the square root of $9x^2-42xy+49y^2$.

$$\begin{array}{l|l}
 & 9x^2-42xy+49y^2\ (\ 3x-7y \\
 & 9x^2 \\
\hline
6x-7y & -42xy+49y^2 \\
 & -42xy+49y^2 \\
\hline
\end{array}$$

Explanation. The square root of $9x^2$ is $3x$, and this is the first term of the root.

By doubling this we obtain $6x$, which is the first term of the divisor. Dividing $-42xy$, the first term of the remainder, by $6x$ we get $-7y$, the new term in the root, which has to be annexed both to the root and divisor. We next multiply the complete divisor by $-7y$ and subtract the result from the first remainder. There is now no remainder, and the root has been found.

The rule can be extended so as to find the square root of any multinomial. The first two terms of the root will be obtained as before. When we have brought down *the second remainder*, the first part of the new divisor is obtained by doubling the terms of the root already found. We then divide the first term of the remainder by the first term of the new divisor, and set down the result as the next term in the root and in the divisor. We next multiply the complete divisor by the last term of the root and subtract the product from the last remainder. If there is now no remainder the root has been found; if there is a remainder we continue the process.

Example 2. Find the square root of

$$25x^2a^2-12xa^3+16x^4+4a^4-24x^3a.$$

Rearrange in descending powers of x.

$$\begin{array}{l|l}
 & 16x^4-24x^3a+25x^2a^2-12xa^3+4a^4\ (\ 4x^2-3xa+2a^2 \\
 & 16x^4 \\
\hline
8x^2-3xa & -24x^3a+25x^2a^2 \\
 & -24x^3a+\ \ 9x^2a^2 \\
\hline
8x^2-6xa+2a^2 & \qquad 16x^2a^2-12xa^3+4a^4 \\
 & \qquad 16x^2a^2-12xa^3+4a^4 \\
\hline
\end{array}$$

Explanation. When we have obtained two terms in the root, $4x^2-3xa$, we have a remainder

$$16x^2a^2-12xa^3+4a^4.$$

Doubling the terms of the root already found, we place the result, $8x^2-6xa$, as the first part of the divisor. Dividing $16x^2a^2$, the first term of the remainder, by $8x^2$, the first term of the divisor, we get $+2a^2$, which we annex both to the root and divisor. We now multiply the complete divisor by $2a^2$ and subtract. There is no remainder, and the root is found.

EXAMPLES XVI. b.

Find the square root of each of the following expressions:

1. $x^2+4xy+4y^2$.

2. $9a^2+12ab+4b^2$.

3. $x^2-10xy+25y^2$.

4. $4x^2-12xy+9y^2$.

5. $81x^2+18xy+y^2$.

6. $25x^2-30xy+9y^2$.

7. $x^4-2x^2y^2+y^4$.

8. $1-2a^3+a^6$.

9. $a^4-2a^3+3a^2-2a+1$.

10. $4x^4-12x^3+29x^2-30x+25$.

11. $9x^4-12x^3-2x^2+4x+1$.

12. $x^4-4x^3+6x^2-4x+1$.

13. $4a^4+4a^3-7a^2-4a+4$.

14. $1-10x+27x^2-10x^3+x^4$.

15. $4x^2+9y^2+25z^2+12xy-30yz-20xz$.

16. $16x^6+16x^7-4x^8-4x^9+x^{10}$.

17. $x^6-22x^4+34x^3+121x^2-374x+289$.

18. $25x^4-30ax^3+49a^2x^2-24a^3x+16a^4$.

19. $4x^4+4x^2y^2-12x^2z^2+y^4-6y^2z^2+9z^4$.

20. $6ab^2c-4a^2bc+a^2b^2+4a^2c^2+9b^2c^2-12abc^2$.

21. $-6b^2c^2+9c^4+b^4-12c^2a^2+4a^4+4a^2b^2$.

22. $4x^4+9y^4+13x^2y^2-6xy^3-4x^3y$.

23. $67x^2+49+9x^4-70x-30x^3$.

24. $1-4x+10x^2-20x^3+25x^4-24x^5+16x^6$.

25. $6acx^5+4b^2x^4+a^2x^{10}+9c^2-12bcx^2-4abx^7$.

[*If preferred, the remainder of this chapter may be postponed and taken after* Chap. XXIV.]

***120.** When the expression whose root is required contains fractional terms, we may proceed as before, the fractional part of the work being performed by the rules explained in Chap. XII.

***121.** There is one important point to be observed when an expression contains powers of a certain letter and also powers of its reciprocal. Thus in the expression

$$2x+\frac{1}{x^2}+4+x^3+\frac{5}{x}+7x^2+\frac{8}{x^3},$$

the order of *descending* powers is

$$x^3+7x^2+2x+4+\frac{5}{x}+\frac{1}{x^2}+\frac{8}{x^3};$$

and the numerical quantity 4 stands between x and $\frac{1}{x}$.

The reason for this arrangement will appear in Chap. XXX.

Example. Find the square root of $24+\frac{16y^2}{x^2}-\frac{8x}{y}+\frac{x^2}{y^2}-\frac{32y}{x}$.

Arrange the expression in descending powers of y.

$$
\begin{array}{l|l}
 & \frac{16y^2}{x^2}-\frac{32y}{x}+24-\frac{8x}{y}+\frac{x^2}{y^2}\left(\frac{4y}{x}-4+\frac{x}{y}\right. \\
 & \frac{16y^2}{x^2} \\ \hline
\frac{8y}{x}-4 & -\frac{32y}{x}+24 \\
 & -\frac{32y}{x}+16 \\ \hline
\frac{8y}{x}-8+\frac{x}{y} & 8-\frac{8x}{y}+\frac{x^2}{y^2} \\
 & 8-\frac{8x}{y}+\frac{x^2}{y^2} \\ \hline
\end{array}
$$

Here the second term in the root, -4, arises from division of $-\frac{32y}{x}$ by $\frac{8y}{x}$, and the third term, $\frac{x}{y}$, arises from division of 8 by $\frac{8y}{x}$; thus $8\div\frac{8y}{x}=8\times\frac{x}{8y}=\frac{x}{y}$.

*EXAMPLES XVI. c.

Find the square root of each of the following expressions :

1. $\frac{x^2}{4}-3x+9$.
2. $4-\frac{4x}{y}+\frac{x^2}{y^2}$.
3. $\frac{x^2}{25}+\frac{2xy}{5}+y^2$.
4. $\frac{x^2}{y^2}+\frac{10x}{y}+25$.
5. $\frac{x^2}{4y^2}-\frac{2x}{y}+4$.
6. $\frac{x^2}{y^2}-\frac{2ax}{by}+\frac{a^2}{b^2}$.
7. $\frac{64x^2}{9y^2}+\frac{32x}{3y}+4$.
8. $\frac{9x^2}{25}-2+\frac{25}{9x^2}$.
9. $\frac{a^4}{64}+\frac{a^3}{8}-a+1$.
10. $x^4+2x^3-x+\frac{1}{4}$.
11. $-3a^3+\frac{25}{9}+a^4-5a+\frac{67}{12}a^2$.
12. $x^4-2x+\frac{1}{9}+\frac{29}{3}x^2-6x^3$.
13. $\frac{a^4}{4}+\frac{a^3}{x}+\frac{a^2}{x^2}-ax-2+\frac{x^2}{a^2}$.
14. $x^4-2x^3+\frac{3x^2}{2}-\frac{x}{2}+\frac{1}{16}$.
15. $\frac{x^4}{4}+4x^2+\frac{ax^2}{3}+\frac{a^2}{9}-2x^3-\frac{4ax}{3}$.

Find the square root of each of the following expressions:

16. $\frac{9a^2}{x^2} - \frac{6a}{5x} + \frac{101}{25} - \frac{4x}{15a} + \frac{4x^2}{9a^2}$.

17. $16m^4 + \frac{16}{3}m^2n + 8m^2 + \frac{4}{9}n^2 + \frac{4}{3}n + 1$.

18. $4x^4 + 32x^2 + 96 + \frac{64}{x^4} + \frac{128}{x^2}$.

***122.** *To find the cube root of a compound expression.*

Since the cube of $a+b$ is $a^3+3a^2b+3ab^2+b^3$, we have to discover a process by which a and b, the terms of the root, can be found when $a^3+3a^2b+3ab^2+b^3$ is given.

The first term a is the cube root of a^3.

Arrange the terms according to powers of one letter a; then the first term is a^3, and its cube root a. Set this down as the first term of the required root. Subtract a^3 from the given expression and the remainder is

$$3a^2b+3ab^2+b^3 \text{ or } (3a^2+3ab+b^2)\times b.$$

Thus b, the second term of the root, will be the quotient when the remainder is divided by $3a^2+3ab+b^2$.

This divisor consists of three terms:

(1) Three times the square of a, the term of the root already found.

(2) Three times the product of the first term a, and the new term b.

(3) The square of b.

The work may be arranged as follows:

$$\begin{array}{lll|l}
 & & & a^3+3a^2b+3ab^2+b^3\,(\,a+b \\
 & & & a^3 \\ \hline
3(a)^2 & = 3a^2 & & 3a^2b+3ab^2+b^3 \\
3\times a\times b = & \quad +3ab & & \\
(b)^2 & = \quad\quad +b^2 & & \\ \cline{2-2}
 & 3a^2+3ab+b^2 & & 3a^2b+3ab^2+b^3
\end{array}$$

Example 1. Find the cube root of $8x^3-36x^2y+54xy^2-27y^3$.

$$\begin{array}{ll|l}
 & & 8x^3-36x^2y+54xy^2-27y^3\,(\,2x-3y \\
 & & 8x^3 \\ \hline
3(2x)^2 & = 12x^2 & -36x^2y+54xy^2-27y^3 \\
3\times 2x\times(-3y) = & \quad -18xy & \\
(-3y)^2 & = \quad\quad +9y^2 & \\ \cline{2-2}
 & 12x^2-18xy+9y^2 & -36x^2y+54xy^2-27y^3
\end{array}$$

Example 2. Find the cube root of $27 + 108x + 90x^2 - 80x^3 - 60x^4 + 48x^5 - 8x^6$.

$$27 + 108x + 90x^2 - 80x^3 - 60x^4 + 48x^5 - 8x^6 \,(3 + 4x - 2x^2$$

$$27$$

$$\begin{aligned}
3 \times (3)^2 &= 27 \\
3 \times 3 \times 4x &= \quad +36x \\
(4x)^2 &= \qquad\quad +16x^2 \\
\hline
& \quad 27 + 36x + 16x^2
\end{aligned}
\qquad
\begin{aligned}
& 108x + 90x^2 - 80x^3 \\
& 108x + 144x^2 + 64x^3 \\
\hline
\end{aligned}$$

$$\begin{aligned}
3 \times (3 + 4x)^2 &= 27 + 72x + 48x^2 \\
3 \times (3 + 4x) \times (-2x^2) &= \qquad\quad - 18x^2 - 24x^3 \\
(-2x^2)^2 &= \qquad\qquad\qquad\quad + 4x^4 \\
\hline
& \quad 27 + 72x + 30x^2 - 24x^3 + 4x^4
\end{aligned}
\qquad
\begin{aligned}
& -54x^2 - 144x^3 - 60x^4 + 48x^5 - 8x^6 \\
& -54x^2 - 144x^3 - 60x^4 + 48x^5 - 8x^6 \\
\hline
\end{aligned}$$

Explanation. When we have obtained two terms in the root, $3 + 4x$, we have a remainder

$$-54x^2 - 144x^3 - 60x^4 + 48x^5 - 8x^6.$$

Take 3 times the square of the root already found and place the result, $27 + 72x + 48x^2$, as the first part of the new divisor. Divide $-54x^2$, the first term of the remainder, by 27, the first term of the divisor; this gives a new term of the root, $-2x^2$. To complete the divisor we take 3 times the product of $(3 + 4x)$ and $-2x^2$, and also the square of $-2x^2$. Now multiply the complete divisor by $-2x^2$ and subtract; there is no remainder and the root is found.

*EXAMPLES XVI. d.

Find the cube root of each of the following expressions :

1. a^3+3a^2+3a+1.

2. $x^3+6x^2+12x+8$.

3. $a^3x^3-3a^2x^2y^2+3axy^4-y^6$.

4. $8m^3-12m^2+6m-1$.

5. $64a^3-144a^2b+108ab^2-27b^3$.

6. $1+3x+6x^2+7x^3+6x^4+3x^5+x^6$.

7. $1-6x+21x^2-44x^3+63x^4-54x^5+27x^6$.

8. $a^3+6a^2b-3a^2c+12ab^2-12abc+3ac^2+8b^3-12b^2c+6bc^2-c^3$.

9. $8a^6-36a^5+66a^4-63a^3+33a^2-9a+1$.

10. $y^6-3y^5+6y^4-7y^3+6y^2-3y+1$.

11. $8x^6+12x^5-30x^4-35x^3+45x^2+27x-27$.

12. $27x^6-54x^5a+117x^4a^2-116x^3a^3+117x^2a^4-54xa^5+27a^6$.

13. $27x^6-27x^5-18x^4+17x^3+6x^2-3x-1$.

14. $24x^4y^2+96x^2y^4-6x^5y+x^6-96xy^5+64y^6-56x^3y^3$.

15. $216+342x^2+171x^4+27x^6-27x^5-109x^3-108x$.

*123. We add some examples of cube root where fractional terms occur in the given expressions.

Example. Find the cube root of $54-27x^3+\dfrac{8}{x^6}-\dfrac{36}{x^3}$.

Arrange the expression in *ascending* powers of x.

$$\frac{8}{x^6}-\frac{36}{x^3}+54-27x^3\left(\frac{2}{x^2}-3x\right.$$

$$\frac{8}{x^6}$$

$$\begin{aligned}
3\times\left(\frac{2}{x^2}\right)^2 &= \frac{12}{x^4} && \qquad -\frac{36}{x^3}+54-27x^3\\
3\times\frac{2}{x^2}\times(-3x) &= -\frac{18}{x}\\
(-3x)^2 &= +9x^2\\
&\quad \frac{12}{x^4}-\frac{18}{x}+9x^2 && \qquad -\frac{36}{x^3}+54-27x^3
\end{aligned}$$

*EXAMPLES XVI. e.

Find the cube root of each of the following expressions:

1. $\frac{x^3}{8} - \frac{3x^2}{4} + \frac{3x}{2} - 1.$

2. $\frac{x^3}{27} + \frac{2x^2}{3} + 4x + 8.$

3. $8x^3 - 4x^2y^2 + \frac{2}{3}xy^4 - \frac{y^6}{27}.$

4. $\frac{27x^3}{64y^3} - \frac{27x^2}{8y^2} + \frac{9x}{y} - 8.$

5. $x^3 - 9x + \frac{27}{x} - \frac{27}{x^3}.$

6. $\frac{x^6}{y^3} - 6x^4 + 12x^2y^3 - 8y^6.$

7. $\frac{x^3}{y^3} + \frac{6x^2}{y^2} + \frac{9x}{y} - 4 - \frac{9y}{x} + \frac{6y^2}{x^2} - \frac{y^3}{x^3}.$

8. $\frac{x^3}{27} - \frac{x^2}{3} + 2x - 7 + \frac{18}{x} - \frac{27}{x^2} + \frac{27}{x^3}.$

9. $\frac{x^3}{a^3} - \frac{12x^2}{a^2} + \frac{54x}{a} - 112 + \frac{108a}{x} - \frac{48a^2}{x^2} + \frac{8a^3}{x^3}.$

10. $\frac{64a^3}{x^3} - \frac{192a^2}{x^2} + \frac{240a}{x} - 160 + \frac{60x}{a} - \frac{12x^2}{a^2} + \frac{x^3}{a^3}.$

11. $\frac{6b}{a} + \frac{6a}{b} - 7 + \frac{a^3}{b^3} - \frac{3a^2}{b^2} - \frac{3b^2}{a^2} + \frac{b^3}{a^3}.$

12. $\frac{60x^4}{y^4} - \frac{80x^3}{y^3} - \frac{90x^2}{y^2} + \frac{8x^6}{y^6} + \frac{108x}{y} - 27 + \frac{48x^5}{y^5}.$

*124. The ordinary rules for extracting square and cube roots in Arithmetic are based upon the algebraical methods explained in the present chapter. The following example is given to illustrate the arithmetical process.

Example. Find the cube root of 614125.

Since 614125 lies between 512000 and 729000, that is, between $(80)^3$ and $(90)^3$, its cube root lies between 80 and 90 and therefore consists of two figures.

$$\begin{array}{rr|l}
 & & \; a+b \\
 & & 614125\,(\,80+5=85 \\
 & & 512000 \\
\hline
3a^2 = 3\times(80)^2 = & 19200 & 102125 \\
3\times a\times b = 3\times 80\times 5 = & 1200 & \\
b^2 = \qquad 5\times 5 = & 25 & \\
\cline{2-2}
 & 20425 & 102125 \\
\cline{3-3}
\end{array}$$

In Arithmetic the ciphers are usually omitted, and there are other modifications of the algebraical rules.

CHAPTER XVII.

RESOLUTION INTO FACTORS.

125. DEFINITION. When an algebraical expression is the product of two or more expressions each of these latter quantities is called a **factor** of it, and the determination of these quantities is called the **resolution** of the expression into its factors.

In this chapter we shall explain the principal rules by which the resolution of expressions into their component factors may be effected.

126. *When each of the terms which compose an expression is divisible by a common factor*, the expression may be simplified by dividing each term separately by this factor, and enclosing the quotient within brackets; the common factor being placed outside as a coefficient.

Example 1. The terms of the expression $3a^2 - 6ab$ have a common factor $3a$;

$$\therefore\ 3a^2 - 6ab = 3a(a - 2b).$$

Example 2. $5a^2bx^3 - 15abx^2 - 20b^3x^2 = 5bx^2(a^2x - 3a - 4b^2)$.

EXAMPLES XVII. a.

Resolve into factors:

1. $a^3 - ax$. **2.** $x^3 - x^2$. **3.** $2a - 2a^2$.

4. $a^2 - ab^2$. **5.** $7p^2 + p$. **6.** $8x - 2x^2$.

7. $5ax - 5a^3x^2$. **8.** $3x^2 + x^5$. **9.** $x^2 + xy$.

10. $x^3 - x^2y$. **11.** $5x - 25x^2y$. **12.** $15 + 25x^2$.

13. $16x + 64x^2y$. **14.** $15a^2 - 225a^4$. **15.** $54 - 81x$.

16. $10x^3 - 25x^4y$. **17.** $3x^3 - x^2 + x$. **18.** $6x^3 + 2x^4 + 4x^5$.

19. $x^3 - x^2y + xy^2$. **20.** $3a^4 - 3a^3b + 6a^2b^2$. **21.** $2x^2y^3 - 6x^2y^2 + 2xy^3$.

22. $6x^3 - 9x^2y + 12xy^2$. **23.** $5x^5 - 10a^2x^3 - 15a^3x^3$.

24. $7a - 7a^3 + 14a^4$. **25.** $38a^3x^5 + 57a^4x^2$.

127. An expression may be resolved into factors *if the terms can be arranged in groups which have a compound factor common.*

Example 1. Resolve into factors $x^2 - ax + bx - ab$.

Noticing that the first two terms contain a common factor x, and the last two terms a common factor b, we enclose the first two terms in one bracket, and the last two in another. Thus

$$\begin{aligned} x^2 - ax + bx - ab &= (x^2 - ax) + (bx - ab) \\ &= x(x-a) + b(x-a) \\ &= (x-a) \text{ taken } x \text{ times } \textit{plus } (x-a) \text{ taken } b \text{ times} \\ &= (x-a) \text{ taken } (x+b) \text{ times} \\ &= (x-a)(x+b). \end{aligned}$$

Example 2. Resolve into factors $6x^2 - 9ax + 4bx - 6ab$.

$$\begin{aligned} 6x^2 - 9ax + 4bx - 6ab &= (6x^2 - 9ax) + (4bx - 6ab) \\ &= 3x(2x - 3a) + 2b(2x - 3a) \\ &= (2x - 3a)(3x + 2b). \end{aligned}$$

Example 3. Resolve into factors $12a^2 - 4ab - 3ax^2 + bx^2$.

$$\begin{aligned} 12a^2 - 4ab - 3ax^2 + bx^2 &= (12a^2 - 4ab) - (3ax^2 - bx^2) \\ &= 4a(3a - b) - x^2(3a - b) \\ &= (3a - b)(4a - x^2). \end{aligned}$$

Note. In the first line of work it is sufficient to see that each pair contains some common factor. Thus, in the last example, by a different arrangement, we have

$$\begin{aligned} 12a^2 - 4ab - 3ax^2 + bx^2 &= (12a^2 - 3ax^2) - (4ab - bx^2) \\ &= 3a(4a - x^2) - b(4a - x^2) \\ &= (4a - x^2)(3a - b), \end{aligned}$$

the same result as before, since it is immaterial in what order the factors of a product are written.

EXAMPLES XVII. b.

Resolve into factors :

1. $a^2 + ab + ac + bc$.
2. $a^2 - ac + ab - bc$.
3. $a^2c^2 + acd + abc + bd$.
4. $a^2 + 3a + ac + 3c$.
5. $2x + cx + 2c + c^2$.
6. $x^2 - ax + 5x - 5a$.
7. $5a + ab + 5b + b^2$.
8. $ab - by - ay + y^2$.
9. $ax - bx - az + bz$.
10. $pr + qr - ps - qs$.
11. $mx - my - nx + ny$.
12. $mx - ma + nx - na$.
13. $2ax + ay + 2bx + by$.
14. $3ax - bx - 3ay + by$.

Resolve into factors :

15. $6x^2+3xy-2ax-ay$.
16. $mx-2my-nx+2ny$.
17. $ax^2-3bxy-axy+3by^2$.
18. $x^2+mxy-4xy-4my^2$.
19. $ax^2+bx^2+2a+2b$.
20. $x^2-3x-xy+3y$.
21. $2x^4-x^3+4x-2$.
22. $3x^3+5x^2+3x+5$.
23. x^4+x^3+2x+2.
24. y^3-y^2+y-1.
25. $axy+bcxy-az-bcz$.
26. $f^2x^2+g^2x^2-ag^2-af^2$.
27. $2ax^2+3axy-2bxy-3by^2$.
28. $amx^2+bmxy-anxy-bny^2$.
29. $ax-bx+by+cy-cx-ay$.
30. $a^2x+abx+ac+aby+b^2y+bc$.

Trinomial Expressions.

128. Before proceeding to the next case of resolution into factors the student is advised to refer to Chap. v. Art. 44. Attention has there been drawn to the way in which, in forming the product of two binomials, the coefficients of the different terms combine so as to give a trinomial result. Thus, by Art. 44,

$$(x+5)(x+3)=x^2+8x+15 \quad\text{.....................}(1),$$
$$(x-5)(x-3)=x^2-8x+15 \quad\text{.....................}(2),$$
$$(x+5)(x-3)=x^2+2x-15 \quad\text{.....................}(3),$$
$$(x-5)(x+3)=x^2-2x-15 \quad\text{.....................}(4).$$

We now propose to consider the converse problem : namely, the resolution of a trinomial expression, similar to those which occur on the right-hand side of the above identities, into its component binomial factors.

By examining the above results, we notice that :

1. The first term of both the factors is x.

2. The *product* of the second terms of the two factors is equal to the *third term* of the trinomial ; e.g. in (2) above we see that 15 is the product of -5 and -3; while in (3) -15 is the product of $+5$ and -3.

3. The *algebraic sum* of the second terms of the two factors is equal to the *coefficient* of x in the trinomial ; e.g. in (4) the sum of -5 and $+3$ gives -2, the coefficient of x in the trinomial.

In applying these laws we will first consider a case where the *third term of the trinomial is positive.*

Example 1. Resolve into factors $x^2+11x+24$.

The second terms of the factors must be such that their product is $+24$, and their sum $+11$. It is clear that they must be $+8$ and $+3$.

$$\therefore\ x^2+11x+24=(x+8)(x+3).$$

Example 2. Resolve into factors $x^2 - 10x + 24$.

The second terms of the factors must be such that their product is $+24$, and their sum -10. Hence they must *both* be *negative*, and it is easy to see that they must be -6 and -4.

$$\therefore\ x^2 - 10x + 24 = (x-6)(x-4).$$

Example 3.
$$\begin{aligned} x^2 - 18x + 81 &= (x-9)(x-9) \\ &= (x-9)^2. \end{aligned}$$

Example 4.
$$\begin{aligned} x^4 + 10x^2 + 25 &= (x^2+5)(x^2+5) \\ &= (x^2+5)^2. \end{aligned}$$

Example 5. Resolve into factors $x^2 - 11ax + 10a^2$.

The second terms of the factors must be such that their product is $+10a^2$, and their sum $-11a$. Hence they must be $-10a$ and $-a$.

$$\therefore\ x^2 - 11ax + 10a^2 = (x-10a)(x-a).$$

Note. In examples of this kind the student should always verify his results, by forming the product (*mentally*, as explained in Chap. v.) of the factors he has chosen.

EXAMPLES XVII. c.

Resolve into factors :

1. $a^2 + 3a + 2$. **2.** $a^2 + 2a + 1$. **3.** $a^2 + 7a + 12$.

4. $a^2 - 7a + 12$. **5.** $x^2 - 11x + 30$. **6.** $x^2 - 15x + 56$.

7. $x^2 - 19x + 90$. **8.** $x^2 + 13x + 42$. **9.** $x^2 - 21x + 110$.

10. $x^2 - 21x + 108$. **11.** $x^2 - 21x + 80$. **12.** $x^2 + 21x + 90$.

13. $x^2 - 19x + 84$. **14.** $x^2 - 19x + 78$. **15.** $x^2 - 18x + 45$.

16. $x^2 + 20x + 96$. **17.** $x^2 - 26x + 165$. **18.** $x^2 - 21x + 104$.

19. $x^2 + 23x + 102$. **20.** $a^2 - 24a + 95$. **21.** $a^2 - 32a + 256$.

22. $a^2 + 30a + 225$. **23.** $a^2 + 54a + 729$. **24.** $a^2 \quad 38a + 361$.

25. $a^2 - 14ab + 49b^2$. **26.** $a^2 + 5ab + 6b^2$.

27. $m^2 - 13mn + 40n^2$. **28.** $m^2 - 22mn + 105n^2$.

29. $x^2 - 23xy + 132y^2$. **30.** $x^2 - 26xy + 169y^2$.

31. $x^4 + 8x^2 + 7$. **32.** $x^4 + 9x^2y^2 + 14y^4$.

33. $x^2y^2 - 16xy + 39$. **34.** $x^2 + 49xy + 600y^2$.

35. $x^2y^2 + 34xy + 289$. **36.** $a^4b^4 + 37a^2b^2 + 300$.

37. $a^2 - 20abx + 75b^2x^2$. **38.** $x^2 + 43xy + 390y^2$.

39. $a^2 - 29ab + 54b^2$. **40.** $x^4 + 162x^2 + 6561$.

41. $12 - 7x + x^2$. **42.** $20 + 9x + x^2$.

43. $132 - 23x + x^2$. **44.** $88 + 19x + x^2$.

45. $130 + 31xy + x^2y^2$. **46.** $143 - 24xa + x^2a^2$.

47. $204 - 29x^2 + x^4$. **48.** $216 + 35x + x^2$.

129. Next consider a case where *the third term of the trinomial is negative.*

Example 1. Resolve into factors $x^2+2x-35$.

The second terms of the factors must be such that their product is -35, and their *algebraical sum* $+2$. Hence they must have *opposite* signs, and the greater of them must be *positive* in order to give its sign to their sum.

The required terms are therefore $+7$ and -5.

$$\therefore\ x^2+2x-35=(x+7)(x-5).$$

Example 2. Resolve into factors $x^2-3x-54$.

The second terms of the factors must be such that their product is -54, and their *algebraical sum* -3. Hence they must have *opposite* signs, and the greater of them must be *negative* in order to give its sign to their sum.

The required terms are therefore -9 and $+6$.

$$\therefore\ x^2-3x-54=(x-9)(x+6).$$

Remembering that in these cases the numerical quantities *must have opposite signs*, if preferred, the following method may be adopted.

Example 3. Resolve into factors $x^2y^2+23xy-420$.

Find two numbers whose product is 420, and whose *difference* is 23. These are 35 and 12; hence inserting the signs so that the positive may predominate, we have

$$x^2y^2+23xy-420=(xy+35)(xy-12).$$

EXAMPLES XVII. d.

Resolve into factors :

1. x^2-x-2.	**2.** x^2+x-2.	**3.** x^2-x-6.
4. x^2+x-6.	**5.** x^2-2x-3.	**6.** x^2+2x-3.
7. x^2+x-56.	**8.** $x^2+3x-40$.	**9.** $x^2-4x-12$.
10. a^2-a-20.	**11.** $a^2-4a-21$.	**12.** a^2+a-20.
13. $a^2-4a-117$.	**14.** $x^2+9x-36$.	**15.** $x^2+x-156$.
16. $x^2+x-110$.	**17.** $x^2-9x-90$.	**18.** $x^2-x-240$.
19. $a^2-12a-85$.	**20.** $a^2-11a-152$.	**21.** $x^2y^2-5xy-24$.
22. $x^2+7xy-60y^2$.	**23.** $x^2+ax-42a^2$.	**24.** $x^2-32xy-105y^2$.
25. $a^2-ay-210y^2$.	**26.** $x^2+18x-115$.	**27.** $x^2-20xy-96y^2$.
28. $x^2+16x-260$.	**29.** $a^2-11a-26$.	**30.** $a^2y^2+14ay-240$.

31. $a^4 - a^2b^2 - 56b^4$. **32.** $x^4 - 14x^2 - 51$. **33.** $y^4 + 6x^2y^2 - 27x^4$.

34. $a^2b^2 - 3abc - 10c^2$. **35.** $a^2 + 12abx - 28b^2x^2$.

36. $a^2 - 18axy - 243x^2y^2$. **37.** $x^4 + 13a^2x^2 - 300a^4$.

38. $x^4 - a^2x^2 - 132a^4$. **39.** $x^4 - a^2x^2 - 462a^4$.

40. $x^6 + x^3 - 870$. **41.** $2 + x - x^2$. **42.** $6 + x - x^2$.

43. $110 - x - x^2$. **44.** $380 - x - x^2$. **45.** $120 - 7ax - a^2x^2$.

46. $65 + 8xy - x^2y^2$. **47.** $98 - 7x - x^2$. **48.** $204 - 5x - x^2$.

[*For easy Miscellaneous Examples see page* 124_A.]

130. We proceed now to the resolution into factors of trinomial expressions when *the coefficient of the highest power is not unity.*

Again, referring to Chap. v. Art. 44, we may write down the following results:

$$(3x+2)(x+4) = 3x^2 + 14x + 8 \quad\ldots\ldots\ldots\ldots(1),$$
$$(3x-2)(x-4) = 3x^2 - 14x + 8 \quad\ldots\ldots\ldots\ldots(2),$$
$$(3x+2)(x-4) = 3x^2 - 10x - 8 \quad\ldots\ldots\ldots\ldots(3),$$
$$(3x-2)(x+4) = 3x^2 + 10x - 8 \quad\ldots\ldots\ldots\ldots(4).$$

The converse problem presents more difficulty than the cases we have yet considered.

Before endeavouring to give a general method of procedure, it will be worth while to examine in detail two of the identities given above.

Consider the result $3x^2 - 14x + 8 = (3x-2)(x-4)$.

The first term $3x^2$ is the product of $3x$ and x.
The third term $+8$.................. -2 and -4.

The middle term $-14x$ is the result of adding together the two products $3x \times -4$ and $x \times -2$.

Again, consider the result $3x^2 - 10x - 8 = (3x+2)(x-4)$.

The first term $3x^2$ is the product of $3x$ and x.
The third term -8.................. $+2$ and -4.

The middle term $-10x$ is the result of adding together the two products $3x \times -4$ and $x \times 2$; and its sign is negative because the greater of these two products is negative.

131. The beginner will frequently find that it is not easy to select the proper factors at the first trial. Practice alone will enable him to detect at a glance whether any pair he has chosen will combine so as to give the correct coefficients of the expression to be resolved.

Example. Resolve into factors $7x^2 - 19x - 6$.

Write down $(7x \quad 3)(x \quad 2)$ for a first trial, noticing that 3 and 2 must have opposite signs. These factors give $7x^2$ and -6 for the first and third terms. But since $7 \times 2 - 3 \times 1 = 11$, the combination fails to give the correct coefficient of the middle term.

Next try $(7x \quad 2)(x \quad 3)$.

Since $7 \times 3 - 2 \times 1 = 19$, these factors will be correct if we insert the signs so that the negative shall predominate.

Thus $$7x^2 - 19x - 6 = (7x + 2)(x - 3).$$

[Verify by mental multiplication.]

132. In actual work it will not be necessary to put down all these steps at length. The student will soon find that the different cases may be rapidly reviewed, and the unsuitable combinations rejected at once.

It is especially important to pay attention to the two following hints:

1. If the third term of the trinomial is positive, then the second terms of its factors have both the same sign, and this sign is the same as that of the middle term of the trinomial.

2. If the third term of the trinomial is negative, then the second terms of its factors have opposite signs.

Example 1. Resolve into factors $14x^2 + 29x - 15$ ....(1),

$14x^2 - 29x - 15$ ....(2).

In each case we may write down $(7x \quad 3)(2x \quad 5)$ as a first trial, noticing that 3 and 5 must have opposite signs.

And since $7 \times 5 - 3 \times 2 = 29$, we have only now to insert the proper signs in each factor.

In (1) the positive sign must predominate,

in 2 the negative ....

Therefore $$14x^2 + 29x - 15 = (7x - 3)(2x + 5).$$
$$14x^2 - 29x - 15 = (7x + 3)(2x - 5).$$

Example 2. Resolve into factors $5x^2 + 17x + 6$ ....(1),

$5x^2 - 17x + 6$ ....(2).

In (1) we notice that the factors which give 6 are both positive.

In (2) .... negative.

And therefore for (1) we may write $(5x + \quad)(x + \quad)$.

(2) .... $(5x - \quad)(x - \quad)$.

And, since $5 \times 3 + 1 \times 2 = 17$, we see that

$$5x^2 + 17x + 6 = (5x + 2)(x + 3).$$
$$5x^2 - 17x + 6 = (5x - 2)(x - 3).$$

Note. In each expression the third term 6 also admits of factors 6 and 1: but this is one of the cases referred to above which the student would reject at once as unsuitable.

Example 3. $9x^2 - 48xy + 64y^2 = (3x - 8y)(3x - 8y)$
$$= (3x - 8y)^2.$$

Example 4. $6 + 7x - 5x^2 = (3 + 5x)(2 - x)$.

EXAMPLES XVII. e.

Resolve into factors:

1. $2x^2+3x+1$.	**2.** $3x^2+5x+2$.	**3.** $2x^2+5x+2$.
4. $3x^2+10x+3$.	**5.** $2x^2+9x+4$.	**6.** $3x^2+8x+4$.
7. $2x^2+7x+6$.	**8.** $2x^2+11x+5$.	**9.** $3x^2+11x+6$.
10. $5x^2+11x+2$.	**11.** $2x^2+3x-2$.	**12.** $3x^2+x-2$.
13. $4x^2+11x-3$.	**14.** $3x^2+14x-5$.	**15.** $2x^2+15x-8$.
16. $2x^2-x-1$.	**17.** $3x^2+7x-6$.	**18.** $2x^2+x-28$.
19. $3x^2+13x-30$.	**20.** $6x^2+7x-3$.	**21.** $6x^2-7x-3$.
22. $3x^2+7x+4$.	**23.** $3x^2+23x+14$.	**24.** $2x^2-x-15$.
25. $3x^2+19x-14$.	**26.** $3x^2-19x-14$.	**27.** $6x^2-31x+35$.
28. $4x^2+x-14$.	**29.** $3x^2-13x+14$.	**30.** $3x^2+41x+26$.
31. $4x^2+23x+15$.	**32.** $2x^2-5xy-3y^2$.	**33.** $8x^2-38x+35$.
34. $12x^2-23xy+10y^2$.	**35.** $15x^2+224x-15$.	**36.** $15x^2-77x+10$.
37. $12x^2-31x-15$.	**38.** $24x^2+22x-21$.	**39.** $72x^2-145x+72$.
40. $24x^2-29xy-4y^2$.	**41.** $2-3x-2x^2$.	**42.** $3+11x-4x^2$.
43. $6+5x-6x^2$.	**44.** $4-5x-6x^2$.	**45.** $5+32x-21x^2$.
46. $7+10x+3x^2$.	**47.** $18-33x+5x^2$.	**48.** $8+6x-5x^2$.
49. $20-9x-20x^2$.	**50.** $24+37x-72x^2$.	

The Difference of Two Squares.

133. By multiplying $a+b$ by $a-b$ we obtain the identity
$$(a+b)(a-b) = a^2 - b^2,$$
a result which may be verbally expressed as follows:

The product of the sum and the difference of any two quantities is equal to the difference of their squares.

Conversely, *the difference of the squares of any two quantities is equal to the product of the sum and the difference of the two quantities.*

Thus any expression which is the difference of two squares may at once be resolved into factors.

Example. Resolve into factors $25x^2 - 16y^2$.

$$25x^2 - 16y^2 = (5x)^2 - (4y)^2.$$

Therefore the first factor is the sum of $5x$ and $4y$,
and the second factor is the difference of $5x$ and $4y$.

$$\therefore\ 25x^2 - 16y^2 = (5x + 4y)(5x - 4y).$$

The intermediate steps may usually be omitted.

Example. $$1 - 49c^6 = (1 + 7c^3)(1 - 7c^3).$$

The difference of the squares of two numerical quantities may be found by the formula $a^2 - b^2 = (a + b)(a - b)$.

Example. $$\begin{aligned}(329)^2 - (171)^2 &= (329 + 171)(329 - 171)\\ &= 500 \times 158\\ &= 79000.\end{aligned}$$

EXAMPLES XVII. f.

Resolve into factors:

1. $x^2 - 4$. **2.** $a^2 - 81$. **3.** $y^2 - 100$.
4. $c^2 - 144$. **5.** $9 - a^2$. **6.** $49 - c^2$.
7. $121 - x^2$. **8.** $400 - a^2$. **9.** $x^2 - 9a^2$.
10. $y^2 - 25x^2$. **11.** $36x^2 - 25b^2$. **12.** $9x^2 - 1$.
13. $36p^2 - 49q^2$. **14.** $4k^2 - 1$. **15.** $49 - 100k^2$.
16. $1 - 25x^2$. **17.** $a^2 - 4b^2$. **18.** $9x^2 - y^2$.
19. $p^2q^2 - 36$. **20.** $a^2b^2 - 4c^2d^2$. **21.** $x^4 - 9$.
22. $9a^4 - 121$. **23.** $25x^2 - 64$. **24.** $81a^4 - 49x^4$.
25. $x^6 - 25$. **26.** $1 - 36a^6$. **27.** $9x^4 - a^2$.
28. $81x^6 - 25a^2$. **29.** $x^4a^2 - 49$. **30.** $a^2 - 64x^6$.
31. $a^2b^2 - 9x^6$. **32.** $x^6y^6 - 4$. **33.** $1 - a^2b^2$.
34. $4 - x^2$. **35.** $9 - 4a^2$. **36.** $9a^4 - 25b^4$.
37. $x^4 - 16b^2$. **38.** $x^2 - 25y^2$. **39.** $1 - 100b^2$.
40. $25 - 64x^2$. **41.** $121a^2 - 81x^2$. **42.** $p^2q^2 - 64a^4$.
43. $64x^2 - 25z^6$. **44.** $49x^4 - 16y^4$. **45.** $81p^4z^6 - 25b^2$.
46. $16x^{16} - 9y^6$. **47.** $36x^{36} - 49a^{14}$. **48.** $1 - 100a^6b^4c^2$.
49. $25x^{10} - 16a^8$. **50.** $a^2b^4c^6 - x^{16}$.

Find by resolving into factors the value of

51. $(575)^2 - (425)^2$. **52.** $(121)^2 - (120)^2$. **53.** $(750)^2 - (250)^2$.
54. $(339)^2 - (319)^2$. **55.** $(753)^2 - (253)^2$. **56.** $(101)^2 - (99)^2$.
57. $(1723)^2 - (277)^2$. **58.** $(1639)^2 - (739)^2$. **59.** $(1811)^2 - (689)^2$
60. $(2731)^2 - (269)^2$. **61.** $(8133)^2 - (8131)^2$. **62.** $(10001)^2 - 1$.

134. When one or both of the squares is a compound quantity the same method is employed.

Example 1. Resolve into factors $(a+2b)^2-16x^2$.

The sum of $a+2b$ and $4x$ is $a+2b+4x$,

and their difference is $a+2b-4x$.

$$\therefore\ (a+2b)^2-16x^2=(a+2b+4x)(a+2b-4x).$$

Example 2. Resolve into factors $x^2-(2b-3c)^2$.

The sum of x and $2b-3c$ is $x+2b-3c$,

and their difference is $x-(2b-3c)=x-2b+3c$.

$$\therefore\ x^2-(2b-3c)^2=(x+2b-3c)(x-2b+3c).$$

If the factors contain like terms they should be collected so as to give the result in its simplest form.

Example 3. $(3x+7y)^2-(2x-3y)^2$

$$\begin{aligned}&=\{(3x+7y)+(2x-3y)\}\{(3x+7y)-(2x-3y)\}\\&=(3x+7y+2x-3y)(3x+7y-2x+3y)\\&=(5x+4y)(x+10y).\end{aligned}$$

EXAMPLES XVII. g.

Resolve into factors :

1. $(a+b)^2-c^2$. **2.** $(a-b)^2-c^2$. **3.** $(x+y)^2-4z^2$.

4. $(x+2y)^2-a^2$. **5.** $(a+3b)^2-16x^2$. **6.** $(x+5a)^2-9y^2$.

7. $(x+5c)^2-1$. **8.** $(a-2x)^2-b^2$. **9.** $(2x-3a)^2-9c^2$.

10. $a^2-(b-c)^2$. **11.** $x^2-(y+z)^2$. **12.** $4a^2-(y-z)^2$.

13. $9x^2-(2a-3b)^2$. **14.** $1-(a-b)^2$. **15.** $c^2-(5a-3b)^2$.

16. $(a+b)^2-(c+d)^2$. **17.** $(a-b)^2-(x+y)^2$. **18.** $(7x+y)^2-1$.

19. $(a+b)^2-(m-n)^2$. **20.** $(a-n)^2-(b+m)^2$. **21.** $(b-c)^2-(a-x)^2$.

22. $(4a+x)^2-(b+y)^2$. **23.** $(a+2b)^2-(3x+4y)^2$.

24. $1-(7a-3b)^2$. **25.** $(a-b)^2-(x-y)^2$.

26. $(a-3x)^2-16y^2$. **27.** $(2a-5x)^2-1$.

28. $(a+b-c)^2-(x-y+z)^2$. **29.** $(3a+2b)^2-(c+x-2y)^2$.

Resolve into factors and simplify :

30. $(x+y)^2-x^2$. **31.** $x^2-(y-x)^2$. **32.** $(x+3y)^2-4y^2$.

33. $(24x+y)^2-(23x-y)^2$. **34.** $(5x+2y)^2-(3x-y)^2$.

35. $9x^2-(3x-5y)^2$. **36.** $(7x+3)^2-(5x-4)^2$.

37. $(3a+1)^2-(2a-1)^2$. **38.** $16a^2-(3a+1)^2$.

39. $(2a+b-c)^2-(a-b+c)^2$. **40.** $(x-7y+z)^2-(7y-z)^2$.

41. $(x+y-8)^2-(x-8)^2$. **42.** $(2x+a-3)^2-(3-2x)^2$.

135. By suitably grouping together the terms, compound expressions can often be expressed as the difference of two squares, and so be resolved into factors.

Example 1. Resolve into factors $a^2-2ax+x^2-4b^2$.

$$\begin{aligned} a^2-2ax+x^2-4b^2 &= (a^2-2ax+x^2)-4b^2 \\ &= (a-x)^2-(2b)^2 \\ &= (a-x+2b)(a-x-2b). \end{aligned}$$

Example 2. Resolve into factors $9a^2-c^2+4cx-4x^2$.

$$\begin{aligned} 9a^2-c^2+4cx-4x^2 &= 9a^2-(c^2-4cx+4x^2) \\ &= (3a)^2-(c-2x)^2 \\ &= (3a+c-2x)(3a-c+2x). \end{aligned}$$

Example 3. Resolve into factors $2bd-a^2-c^2+b^2+d^2+2ac$.

Here the terms $2bd$ and $2ac$ suggest the proper preliminary arrangement of the expression. Thus

$$\begin{aligned} 2bd-a^2-c^2+b^2+d^2+2ac &= b^2+2bd+d^2-a^2+2ac-c^2 \\ &= b^2+2bd+d^2-(a^2-2ac+c^2) \\ &= (b+d)^2-(a-c)^2 \\ &= (b+d+a-c)(b+d-a+c). \end{aligned}$$

Example 4. Resolve into factors $x^4+x^2y^2+y^4$.

$$\begin{aligned} x^4+x^2y^2+y^4 &= (x^4+2x^2y^2+y^4)-x^2y^2 \\ &= (x^2+y^2)^2-(xy)^2 \\ &= (x^2+y^2+xy)(x^2+y^2-xy) \\ &= (x^2+xy+y^2)(x^2-xy+y^2). \end{aligned}$$

This result is very important and will be referred to again in Chapter XXVIII.

EXAMPLES XVII. h.

Resolve into factors :

1. $x^2+2xy+y^2-a^2$.	**2.** $a^2-2ab+b^2-x^2$.
3. $x^2-6ax+9a^2-16b^2$.	**4.** $4a^2+4ab+b^2-9c^2$.
5. $x^2+a^2+2ax-y^2$.	**6.** $2ay+a^2+y^2-x^2$.
7. $x^2-a^2-2ab-b^2$.	**8.** $y^2-c^2+2cx-x^2$.
9. $1-x^2-2xy-y^2$.	**10.** $c^2-x^2-y^2+2xy$.
11. $x^2+y^2+2xy-4x^2y^2$.	**12.** $a^2-4ab+4b^2-9a^2c^2$.
13. $x^2+2xy+y^2-a^2-2ab-b^2$.	**14.** $a^2-2ab+b^2-c^2-2cd-d^2$.
15. $x^2-4ax+4a^2-b^2+2by-y^2$.	**16.** $y^2+2by+b^2-a^2-6ax-9x^2$.

17. $x^2 - 2x + 1 - a^2 - 4ab - 4b^2$. **18.** $9a^2 - 6a + 1 - x^2 - 8dx - 16d^2$.

19. $x^2 - a^2 + y^2 - b^2 - 2xy + 2ab$. **20.** $a^2 + b^2 - 2ab - c^2 - d^2 - 2cd$.

21. $4x^2 - 12ax - c^2 - k^2 - 2ck + 9a^2$.

22. $a^2 + 6bx - 9b^2x^2 - 10ab - 1 + 25b^2$.

23. $a^4 - 25x^6 + 8a^2x^2 - 9 + 30x^3 + 16x^4$.

24. $x^4 - x^2 - 9 - 2a^2x^2 + a^4 + 6x$.

25. $a^4 + a^2b^2 + b^4$. **26.** $x^4 + 4x^2y^2 + 16y^4$. **27.** $p^4 + 9p^2q^2 + 81q^4$.

28. $c^4 + 3c^2d^2 + 4d^4$. **29.** $x^4 + y^4 - 11x^2y^2$. **30.** $4m^4 - 5m^2n^2 + n^4$.

The Sum or Difference of Two Cubes.

136. If we divide $a^3 + b^3$ by $a + b$ the quotient is $a^2 - ab + b^2$; and if we divide $a^3 - b^3$ by $a - b$ the quotient is $a^2 + ab + b^2$.

We have therefore the following identities :

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2) ;$$
$$a^3 - b^3 = (a - b)(a^2 + ab + b^2).$$

These results enable us to resolve into factors any expression which can be written as the sum or the difference of two cubes.

Example 1.
$$8x^3 - 27y^3 = (2x)^3 - (3y)^3$$
$$= (2x - 3y)(4x^2 + 6xy + 9y^2).$$

Note. The middle term $6xy$ is the *product* of $2x$ and $3y$.

Example 2.
$$64a^3 + 1 = (4a)^3 + (1)^3$$
$$= (4a + 1)(16a^2 - 4a + 1).$$

We may usually omit the intermediate step and write down the factors at once.

Examples.
$$343a^6 - 27x^3 = (7a^2 - 3x)(49a^4 + 21a^2x + 9x^2).$$
$$8x^9 + 729 = (2x^3 + 9)(4x^6 - 18x^3 + 81).$$

EXAMPLES XVII. k.

Resolve into factors :

1. $x^3 - y^3$. **2.** $x^3 + y^3$. **3.** $x^3 - 1$. **4.** $1 + a^3$

5. $8x^3 - y^3$. **6.** $x^3 + 8y^3$. **7.** $27x^3 + 1$. **8.** $1 - 8y^3$.

9. $a^3b^3 - c^3$. **10.** $8x^3 + 27y^3$. **11.** $1 - 343x^3$. **12.** $64 + y^3$.

13. $125 + a^3$. **14.** $216 - a^3$. **15.** $a^3b^3 + 512$. **16.** $1000y^3 - 1$.

Resolve into factors :

17. x^3+64y^3. **18.** $27-1000x^3$. **19.** $a^3b^3+216c^3$.

20. $343-8x^3$. **21.** a^3+27b^3. **22.** $27x^3-64y^3$.

23. $125x^3-1$. **24.** $216p^3-343$. **25.** $x^3y^3+z^3$.

26. $a^3b^3c^3-1$. **27.** $343x^3+1000y^3$. **28.** $729a^3-64b^3$.

29. $8a^3b^3+125x^3$. **30.** $x^3y^3-216z^3$. **31.** x^6-27y^3.

32. $64x^6+125y^3$. **33.** $8x^3-z^6$. **34.** $216x^6-b^3$.

35. a^3+343b^3. **36.** a^6+729b^3. **37.** $8x^3-729y^6$.

38. $p^3q^3-27x^3$. **39.** z^3-64y^6. **40.** x^3y^3-512.

136$_A$. In Arts. 128 to 132 we have discussed the factorisation of trinomials by trial. And in Arts. 133 to 135 we have shewn how any expression which is the difference of two squares can be written down as the product of two factors. We shall now explain a general method by which any expression of the form x^2+px+q or ax^2+bx+c can be expressed as the difference of two squares.

By Art. 112 we have the following identities :

$$x^2+2ax+a^2=(x+a)^2, \quad x^2-2ax+a^2=(x-a)^2.$$

So that if a trinomial is a perfect square, and *its highest power x^2 has unity for its coefficient*, we must always have the term without x equal to *the square of half the coefficient of x*. If therefore the first two terms (containing x^2 and x) of such a trinomial are given, the square may be completed by adding the square of half the coefficient of x.

Thus x^2+6x is made a perfect square if we add to it $\left(\frac{6}{2}\right)^2$, or 9 ; and it then becomes x^2+6x+9, or $(x+3)^2$.

Similarly to make x^2-7x a perfect square we must add $\left(-\frac{7}{2}\right)^2$, or $\frac{49}{4}$, and we then have $x^2-7x+\frac{49}{4}$, or $\left(x-\frac{7}{2}\right)^2$.

Note. The added term is always positive.

Example 1. Find the factors of x^2+6x+5.

The expression may be written $(x^2+6x+9)+5-9$;

that is,
$$\begin{aligned} x^2+6x+5&=(x+3)^2-4 \\ &=(x+3+2)(x+3-2) \\ &=(x+5)(x+1). \end{aligned}$$

Example 2. Find the factors of $x^2-7x-228$.

$$x^2-7x-228=\left(x^2-7x+\frac{49}{4}\right)-228-\frac{49}{4}$$
$$=\left(x-\frac{7}{2}\right)^2-\frac{961}{4}$$
$$=\left(x-\frac{7}{2}+\frac{31}{2}\right)\left(x-\frac{7}{2}-\frac{31}{2}\right)$$
$$=(x+12)(x-19).$$

Example 3. Find the factors of $3x^2-13x+14$.

$$3x^2-13x+14=3\left(x^2-\frac{13}{3}x+\frac{14}{3}\right)$$
$$=3\left\{x^2-\frac{13}{3}x+\left(\frac{13}{6}\right)^2+\frac{14}{3}-\frac{169}{36}\right\}$$
$$=3\left\{\left(x-\frac{13}{6}\right)^2-\frac{1}{36}\right\}$$
$$=3\left(x-\frac{13}{6}+\frac{1}{6}\right)\left(x-\frac{13}{6}-\frac{1}{6}\right)$$
$$=3\left(x-\frac{7}{3}\right)(x-2)$$
$$=(3x-7)(x-2).$$

As the process of completing the square is quite general and applicable to all cases, it may conveniently be used when factorisation by trial would prove uncertain and tedious. For example, if the factors of $24x^2+118x-247$ were required, it would probably be best to apply the general method at once.

136_B. The following exercise contains easy miscellaneous examples of the different cases explained in this chapter.

EXAMPLES XVII 1. (Miscellaneous.)

(*On Arts.* 128, 129.)

Resolve into factors:

1. $x^2-3x+2.$ **2.** $a^2+7a+10.$ **3.** $b^2+b-12.$

4. $y^2-4y-21.$ **5.** $c^2+12c+11.$ **6.** $x^2-4x-5.$

7. $n^2+12n+20.$ **8.** $y^2+9y-10.$ **9.** $p^2-2pq-24q^2.$

10. $y^2+y-110.$ **11.** $z^2-9z-90.$ **12.** $k^2-14k+48.$

Resolve into factors:

13. $a^2+18a+81$. 14. $b^2-24b-81$. 15. $c^2+30c+81$.
16. $x^2-14x+49$. 17. $y^2+10yz+21z^2$. 18. $z^2+2z-63$.
19. $n^2+11n+24$. 20. $p^2-5p-24$. 21. $l^2+9l-36$.
22. $a^2b^2-4ab+4$. 23. $a^2b^2+10ab+16$. 24. $b^2-4b-45$.
25. $m^2+3m-88$. 26. $n^2-12n-45$. 27. $p^2+10p-39$.
28. $x^2y^2-xy-72$. 29. z^2-z-20. 30. $x^2+xy-56y^2$.
31. $a^2-11ab-26b^2$. 32. $a^2b^2-ab-56$. 33. y^4+y^2-156.
34. z^4-7z^2-78. 35. y^4-2y^2-35. 36. $x^2+6xy-91y^2$.

(*On Arts.* 125–132.)

Resolve into two or more factors:

37. $m^3n^2-3m^2n^3$. 38. $10x^3+25x^4y$. 39. $y^2-2y-15$.
40. $(a+b)x+(a+b)y$. 41. $x^2-xz+xy-yz$.
42. $3c^2+c-2$. 43. $2b^2+11b+5$. 44. $x^2-6xy+9y^2$.
45. $3x^2-10x+3$. 46. c^2d^2-cd-2. 47. $6x^2+7x-3$.
48. $4(a-b)-c(a-b)$. 49. a^4+a^3+2a+2.
50. $2c^3d-6c^2d^2+2c^2d^3$. 51. $x^3y+2x^2y-63xy$.
52. $6y^2-7y-3$. 53. $4x^2-12x+9$. 54. $3-5p-12p^2$.
55. $16+8pq+p^2q^2$. 56. $4z^3+5z^2-6z$. 57. a^3+a^2-42a.
58. $2m^4-m^3+4m-2$. 59. $a^4-3a^3-a^3b+3a^2b$.
60. $14-5x-x^2$. 61. $17-18z+z^2$. 62. $2m^4-11m^2-21$.
63. $5x^2+7xy-6y^2$. 64. $6m^6+17m^3-45$. 65. $9m^2-24m+16$.

(*On Arts.* 125–136_A.)

66. $25-81a^2$. 67. a^4b^4-9. 68. $27+l^3$. 69. $1-64m^3$.
70. k^4-25l^2. 71. p^3q^3-1. 72. $8z^3+1$. 73. $1-64x^2$.
74. $250p^3+2$. 75. $100a^2b^4-4$. 76. $729+c^3d^3$.
77. $(a+x)^2-1$. 78. $16-(b-c)^2$. 79. $9x^3-4xy^2$.
80. $p^2-pq-20q^2$. 81. l^3-l^2-42l. 82. $a^2b^2c^2-81d^2$.
83. $64x^6-27y^3$. 84. $x^2+2x-323$. 85. x^4-289.
86. $l^2+l-272$. 87. $1000z^3-27$. 88. $a^2+10a-299$.
89. $a^2-b^2-2bc-c^2$. 90. $1-x^2+6xy-9y^2$.
91. $x^4+y^4-7x^2y^2$. 92. a^4+3a^2+4. 93. $b^2-2b-783$.

137. Miscellaneous cases of resolution into factors.

Example 1. Resolve into factors $16a^4 - 81b^4$.

$$16a^4 - 81b^4 = (4a^2 + 9b^2)(4a^2 - 9b^2)$$
$$= (4a^2 + 9b^2)(2a + 3b)(2a - 3b).$$

Example 2. Resolve into factors $x^6 - y^6$.

$$x^6 - y^6 = (x^3 + y^3)(x^3 - y^3)$$
$$= (x + y)(x^2 - xy + y^2)(x - y)(x^2 + xy + y^2).$$

Note. When an expression can be arranged either as the difference of two squares, or as the difference of two cubes, it will be found simplest to first use the rule for the difference of two squares.

Example 3. Resolve into factors $28x^4y + 64x^3y - 60x^2y$.

$$28x^4y + 64x^3y - 60x^2y = 4x^2y(7x^2 + 16x - 15)$$
$$= 4x^2y(7x - 5)(x + 3).$$

Example 4. Resolve into factors $x^3p^2 - 8y^3p^2 - 4x^3q^2 + 32y^3q^2$.

The expression $= p^2(x^3 - 8y^3) - 4q^2(x^3 - 8y^3)$
$$= (x^3 - 8y^3)(p^2 - 4q^2)$$
$$= (x - 2y)(x^2 + 2xy + 4y^2)(p + 2q)(p - 2q).$$

Example 5. Resolve into factors $4x^2 - 25y^2 + 2x + 5y$.

$$4x^2 - 25y^2 + 2x + 5y = (2x + 5y)(2x - 5y) + 2x + 5y$$
$$= (2x + 5y)(2x - 5y + 1).$$

EXAMPLES XVII. 1. (*Continued.*)

Resolve into two or more factors :

94. $x^6 - 64$. **95.** $729y^6 - 64x^6$. **96.** $x^8 - 1$.
97. $729a^7b - ab^7$. **98.** $a^8x^6 - 64a^2y^6$. **99.** $a^{12} - b^{12}$.
100. $x^4 + 4x^2y^2z^2 + 4y^4z^4$. **101.** $a^3b^3 + 512$. **102.** $2x^2 + 17x + 35$.
103. $500x^2y - 20y^3$. **104.** $(a + b)^4 - 1$. **105.** $(c + d)^3 - 1$.
106. $1 - (x - y)^3$. **107.** $x^2 - 6x - 247$. **108.** $a^2 - 22a - 279$.
109. $250(a - b)^3 + 2$. **110.** $(c + d)^3 + (c - d)^3$.
111. $8(x + y)^3 - (2x - y)^3$. **112.** $x^2 - 4y^2 + x - 2y$.
113. $a^2 - b^2 + a - b$. **114.** $(a + b)^2 + a + b$.
115. $a^3 + b^3 + a + b$. **116.** $a^2 - 9b^2 + a + 3b$.
117. $4(x - y)^3 - (x - y)$. **118.** $x^4y - x^2y^3 - x^3y^2 + xy^4$.

[Miscellaneous Examples IV., p. 174, *and* Chapter XXVIII. *will furnish further practice in Resolution into Factors.*]

MISCELLANEOUS EXAMPLES III.

1. Subtract $3x^3-7x+1$ from $2x^2-5x-3$, then subtract the difference from zero, and add this last result to $2x^2-2x^3-4$.

2. Simplify

$$2\{3a-(4b-5c)\}+4\{4a-(5b-2c)\}+4\{5a-3(b-c)\}.$$

3. Find the product of

$$a^3-2a^2c+2ac^2-c^3$$

and

$$a^3+2a^2c+2ac^2+c^3.$$

4. Solve the equations:

(1) $\frac{x}{2}+\frac{x}{3}=\frac{x}{4}+7$; (2) $9x+5y=75$, $7x-4y=11$.

5. Find the square root of $8x^4+16x^2+1-8x-2x^3+x^6$.

6. Find a number whose third, fourth, sixth, and eighth parts together make up 63.

---

7. If $a=4$, $b=3$, $c=2$, find the value of $\frac{a^2-b^2}{b+c}+\frac{b^2-c^2}{c+a}+\frac{c^2-a^2}{a+b}$.

8. Divide $x^4+\frac{9}{4}x^3+\frac{21}{8}x^2+\frac{33}{16}x+\frac{5}{16}$ by $x^2+\frac{3x}{2}+\frac{1}{4}$.

9. Add $5x^2-6x$ to the excess of 1 over $3x^2-5x+1$.

10. Find the factors of (1) $a^2x^2-2ax-15$; (2) $4m^4-81p^2q^2$.

11. Solve the equations:

(1) $13x+11y=18$, $11x+13y=30$. (2) $57x+52y=181$, $76x-39y=458$.

12. A train which travels a miles in b hours is p times as fast as a coach. If the coach takes m hours to cover the distance between two places, how many miles are they apart?

---

13. Find the continued product of $3x^2-2x+3$, $4x+5$, $7x-2$.

14. Solve the equations:

(1) $\frac{5x}{7}-\frac{4}{5}\left(x-\frac{3}{4}\right)-\frac{2}{21}\left(x+\frac{7}{2}\right)+1=0$;

(2) $2\left(\frac{5x}{3}-1\right)+\frac{11}{5}\left(1+\frac{14x}{33}\right)=\frac{2x+7}{5}-7$.

15. Write down the square of $x^2+7x-11$.

16. Resolve into factors:

(1) $x^2+2ax-bx-2ab$; (2) $x^4+10x^2y-56y^2$

17. Find the H.C.F. and L.C.M. of $49bc^3$, $21a^2b^2$, $56ca^3$, $63abc^2$.

18. A has \$50, and B has \$6; after B has received from A a certain sum he then has $\frac{5}{9}$ of what A has: how much did B receive?

---

19. Simplify $\dfrac{15a^2p^3}{56mk^2}\times\dfrac{49ak^5}{40p^2m^3}\div\dfrac{7a^3k^3}{64m^4}$.

20. Shew that $a(a-1)(a-2)(a-3)=(a^2-3a+1)^2-1$.

21. Express by means of symbols:

(1) The excess of m over n is greater than a by c;

(2) Three times the square of ab together with the cube of c is equal to p times the sum of m and n.

22. Solve $\dfrac{x}{4}\left(3-\dfrac{8}{x}\right)-\dfrac{7}{8}\left(7-\dfrac{3x}{4}\right)=15\left(\dfrac{1}{3}-\dfrac{x}{64}\right)$,

and shew that $x=2$ does *not* satisfy the equation.

23. Divide the product of $3x^2-2xy-y^2$ and $2x-y$ by $x-y$.

24. In 9 hours a coach travels one mile more than a train does in 2 hours, but in three hours the train travels two miles more than the coach does in 13 hours; find the rate of each per hour.

---

25. Express the product $(2x^2-13x+15)(x^2-4x-5)(2x^2-x-3)$ in simple factors, and thence write down its square root as the product of three binomial factors.

26. If $x=6$, $y=7$, $z=8$, find the value of

$$x-(y-z)-2[x+z-3\{-2(y-1)\}]+4\left[\frac{x}{2}-\left(3-\frac{9}{2}y\right)\right].$$

27. Divide $6x^5+57x^4y+128x^3y^2-60x^2y^3-130xy^4+63y^5$

by $3x^3+15x^2y+7xy^2-9y^3$.

28. Solve the equations:

$$4x+2y+z=14,\quad 3x-y+2z=3,\quad x+7y-z=23.$$

29. Resolve into two or more factors:

(1) x^3y-4xy^3; (2) $2m^4+m^2n^2-3n^4$.

30. In how many days will a men do $\dfrac{1}{m}$th of a piece of work, the whole of which can be done by b men in c days?

If $m=4$, $a=24$, $b=14$, $c=18$, what is the numerical value of the answer?

CHAPTER XVIII.

HIGHEST COMMON FACTOR.

138. DEFINITION. The **highest common factor** of two or more algebraical expressions is the *expression of highest dimensions* which divides each of them without remainder.

Note. The term *greatest common measure* is sometimes used instead of *highest common factor*; but, strictly speaking, the term *greatest common measure* ought to be confined to arithmetical quantities; for the highest common factor is not necessarily the greatest common measure in all cases, as will appear later. [Art. 145.]

In Chap. XI. we have explained how to write down by inspection the highest common factor of two or more *simple* expressions. An analogous method will enable us readily to find the highest common factor of *compound* expressions which are given as the product of factors, or which can be easily resolved into factors.

Example 1. Find the highest common factor of

$$4cx^3 \text{ and } 2cx^3+4c^2x^2.$$

It will be easy to pick out the common factors if the expressions are arranged as follows:

$$4cx^3=4cx^3,$$

$$2cx^3+4c^2x^2=2cx^2(x+2c);$$

therefore the H.C.F. is $2cx^2$.

Example 2. Find the highest common factor of

$$3a^2+9ab,\ \ a^3-9ab^2,\ \ a^3+6a^2b+9ab^2.$$

Resolving each expression into its factors, we have

$$3a^2+9ab=3a(a+3b),$$

$$a^3-9ab^2=a(a+3b)(a-3b),$$

$$a^3+6a^2b+9ab^2=a(a+3b)(a+3b);$$

therefore the H.C.F. is $a(a+3b)$.

139. When there are two or more expressions containing different powers of the same *compound* factor, the student should be careful to notice that the highest common factor must contain the highest power of the compound factor which is common to all the given expressions.

Example 1. The highest common factor of

$$x(a-x)^2, \ a(a-x)^3, \text{ and } 2ax(a-x)^5 \text{ is } (a-x)^2.$$

Example 2. Find the highest common factor of

$$ax^2+2a^2x+a^3, \ 2ax^2-4a^2x-6a^3, \ 3(ax+a^2)^2.$$

Resolving the expressions into factors, we have

$$ax^2+2a^2x+a^3=a(x^2+2ax+a^2)$$
$$=a(x+a)^2 \quad \ldots\ldots\ldots\ldots(1),$$
$$2ax^2-4a^2x-6a^3=2a(x^2-2ax-3a^2)$$
$$=2a(x+a)(x-3a) \quad \ldots\ldots\ldots\ldots(2),$$
$$3(ax+a^2)^2=3a^2(x+a)^2 \quad \ldots\ldots\ldots\ldots(3).$$

Therefore from (1), (2), (3), by inspection, the highest common factor is $a(x+a)$.

EXAMPLES XVIII. a.

Find the highest common factor of

1. $a^2+ab, \ a^2-b^2.$
2. $(x+y)^2, \ x^2-y^2.$
3. $2x^2-2xy, \ x^3-x^2y.$
4. $6x^2-9xy, \ 4x^2-9y^2.$
5. $x^3+x^2y, \ x^3+y^3.$
6. $a^3b-ab^3, \ a^5b^2-a^2b^5.$
7. $a^3-a^2x, \ a^3-ax^2, \ a^4-ax^3.$
8. $a^2-4x^2, \ a^2+2ax.$
9. $a^2bx+ab^2x, \ a^2b-b^3.$
10. $2x^2y-6xy^2, \ x^2-9y^2.$
11. $a^2-x^2, \ a^2-ax, \ a^2x-ax^2.$
12. $4x^2+2xy, \ 12x^2y-3y^3.$
13. $20x-4, \ 50x^2-2.$
14. $6bx+4by, \ 9cx+6cy.$
15. $x^2+x, \ (x+1)^2, \ x^3+1.$
16. $xy-y, \ x^4y-xy.$
17. $x^2-2xy+y^2, \ (x-y)^3.$
18. $x^3+a^2x, \ x^4-a^4.$
19. $x^3+8y^3, \ x^2+xy-2y^2.$
20. $x^4-27a^3x, \ (x-3a)^2.$
21. $x^2+3x+2, \ x^2-4.$
22. $x^2-x-20, \ x^2-9x+20.$
23. $x^2-18x+45, \ x^2-9.$
24. $2x^2-7x+3, \ 3x^2-7x-6.$
25. $12x^2+x-1, \ 15x^2+8x+1.$
26. $2x^2-x-1, \ 3x^2-x-2.$
27. $c^2x^2-d^2, \ acx^2-bcx+adx-bd.$
28. $x^5-xy^2, \ x^3+x^2y+xy+y^2.$
29. $a^3x-a^2bx-6ab^2x, \ a^2bx^2-4ab^2x^2+3b^3x^2.$
30. $2x^2+9x+4, \ 2x^2+11x+5, \ 2x^2-3x-2.$
31. $3x^4+8x^3+4x^2, \ 3x^5+11x^4+6x^3, \ 3x^4-16x^3-12x^2.$

[*If preferred, the remainder of this chapter may be taken after* Chap. XXV.]

***140.** The highest common factor should always be found by inspection if possible, but it may happen that the expressions cannot be readily resolved into factors. In such cases we adopt a method analogous to that used in Arithmetic, for finding the greatest common measure of two or more numbers.

***141.** We shall now illustrate the algebraical process of finding the highest common factor by examples, postponing for the present the complete proof of the rules we use. But we shall *enunciate* two principles, which the student should bear in mind in reading the examples which follow.

I. *If an expression contains a certain factor, any multiple of the expression is divisible by that factor.*

II. *If two expressions have a common factor, it will divide their sum and their difference; and also the sum and the difference of any multiples of them.*

Example. Find the highest common factor of

$4x^3-3x^2-24x-9$ and $8x^3-2x^2-53x-39$.

$$\begin{array}{r|r|r|l}
x & 4x^3-3x^2-24x-9 & 8x^3-2x^2-53x-39 & 2 \\
 & 4x^3-5x^2-21x & 8x^3-6x^2-48x-18 & \\ \cline{2-3}
2x & 2x^2-3x-9 & 4x^2-5x-21 & 2 \\
 & 2x^2-6x & 4x^2-6x-18 & \\ \cline{2-3}
3 & 3x-9 & x-3 & \\
 & 3x-9 & & \\ \cline{2-2}
\end{array}$$

Therefore the H.C.F. is $x-3$.

Explanation. First arrange the given expressions according to descending or ascending powers of x. The expressions so arranged having their first terms of the same order, we take for divisor that whose highest power has the smaller coefficient. Arrange the work in parallel columns as above. When the first remainder $4x^2-5x-21$ is made the divisor we put the quotient x to the *left* of the dividend. Again, when the second remainder $2x^2-3x-9$ is in turn made the divisor, the quotient 2 is placed to the *right*; and so on. As in Arithmetic, the last divisor $x-3$ is the highest common factor required.

***142.** This method is only useful to determine the *compound* factor of the highest common factor. Simple factors of the given expressions must be first removed from them, and the highest common factor of these, if any, must be observed and multiplied into the *compound* factor given by the rule.

Example. Find the highest common factor of

$$24x^4-2x^3-60x^2-32x \text{ and } 18x^4-6x^3-39x^2-18x.$$

We have $24x^4-2x^3-60x^2-32x=2x(12x^3-x^2-30x-16)$,

and $\qquad 18x^4-6x^3-39x^2-18x=3x(6x^3-2x^2-13x-6)$.

Also $2x$ and $3x$ have the common factor x. Removing the simple factors $2x$ and $3x$, and *reserving* their common factor x, we continue as in Art. 141.

$$\begin{array}{r|r|r|r}
2x & 6x^3-2x^2-13x-6 & 12x^3-x^2-30x-16 & 2 \\
 & 6x^3-8x^2-8x & 12x^3-4x^2-26x-12 & \\ \cline{2-3}
2 & 6x^2-5x-6 & 3x^2-4x-4 & x \\
 & 6x^2-8x-8 & 3x^2+2x & \\ \cline{2-3}
 & 3x+2 & -6x-4 & -2 \\
 & & -6x-4 & \\ \cline{3-3}
\end{array}$$

Therefore the H.C.F. is $x(3x+2)$.

*143. So far the process of Arithmetic has been found exactly applicable to the algebraical expressions we have considered. But in many cases certain modifications of the arithmetical method will be found necessary. These will be more clearly understood if it is remembered that, at every stage of the work, the remainder must contain as a factor of itself the highest common factor we are seeking. [See Art. 141, I. & II.].

Example 1. Find the highest common factor of

$$3x^3-13x^2+23x-21 \text{ and } 6x^3+x^2-44x+21.$$

$$\begin{array}{r|r|l}
3x^3-13x^2+23x-21 & 6x^3+x^2-44x+21 & 2 \\
 & 6x^3-26x^2+46x-42 & \\ \cline{2-2}
 & 27x^2-90x+63 & \\
\end{array}$$

Here on making $27x^2-90x+63$ a divisor, we find that it is not contained in $3x^3-13x^2+23x-21$ with an *integral* quotient. But noticing that $27x^2-90x+63$ may be written in the form $9(3x^2-10x+7)$, and also bearing in mind that every remainder in the course of the work contains the H.C.F., we conclude that the H.C.F. we are seeking is contained in $9(3x^2-10x+7)$. But the two original expressions have no *simple* factors, therefore their H.C.F. can have none. We may therefore *reject* the factor 9 and go on with divisor $3x^2-10x+7$.

Resuming the work, we have

$$\begin{array}{r|r|r|l}
x & 3x^3-13x^2+23x-21 & 3x^2-10x+7 & x \\
 & 3x^3-10x^2+\ \ 7x & 3x^2-\ \ 7x & \\ \hline
-1 & -\ \ 3x^2+16x-21 & -\ \ 3x+7 & -1 \\
 & -\ \ 3x^2+10x-\ \ 7 & -\ \ 3x+7 & \\ \hline
 & 2\,)\,6x-14 & & \\
 & 3x-\ \ 7 & &
\end{array}$$

Therefore the H.C.F. is $3x-7$.

The factor 2 has been removed on the same grounds as the factor 9 above.

Example 2. Find the highest common factor of

$$2x^3+\ x^2-x-2 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and

$$3x^3-2x^2+x-2 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

As the expressions stand we cannot begin to divide one by the other without using a fractional quotient. The difficulty may be obviated by *introducing* a suitable factor, just as in the last case we found it useful to remove a factor when we could no longer proceed with the division in the ordinary way. The given expressions have no common *simple* factor, hence their H.C.F. cannot be affected if we multiply either of them by any simple factor.

Multiply (2) by 2, and use (1) as a divisor:

$$\begin{array}{r|r|r|l}
 & 2x^3+\ \ x^2-\ \ x-\ 2 & 6x^3-\ \ 4x^2+\ 2x-\ 4 & 3 \\
 & 7 & 6x^3+\ \ 3x^2-\ 3x-\ 6 & \\ \hline
-2x & 14x^3+\ 7x^2-\ 7x-14 & -\ \ 7x^2+\ 5x+\ 2 & \\
 & 14x^3-10x^2-\ 4x & 17 & \\ \hline
17x & 17x^2-\ 3x-14 & -119x^2+85x+34 & -7 \\
 & 17x^2-17x & -119x^2+21x+98 & \\ \hline
14 & 14x-14 & 64\,)\,64x-64 & \\
 & 14x-14 & x-\ 1 &
\end{array}$$

Therefore the H.C.F. is $x-1$.

After the first division the factor 7 is introduced because the first remainder $-7x^2+5x+2$ will not divide $2x^3+x^2-x-2$. At the next stage the factor 17 is introduced for a similar reason, and finally the factor 64 is removed as explained in Example 1.

Note. Here the highest common factor might have been more easily obtained by arranging the expressions in *ascending* powers of x. In this case it will be found that there is no need to introduce a numerical factor in the course of the work. Detached coefficients, as explained in Art. 45, may also be used with advantage here, and will often effect a considerable saving of labour.

***144.** From the last two examples it appears that we may multiply or divide either of the given expressions, or any of the remainders which occur in the course of the work, by any factor which does not divide both of the given expressions.

***145.** Let the two expressions in Example 2, Art. 143, be written in the form

$$2x^3+x^2-x-2=(x-1)(2x^2+3x+2),$$
$$3x^3-2x^2+x-2=(x-1)(3x^2+x+2).$$

Then their highest common factor is $x-1$, and therefore $2x^2+3x+2$ and $3x^2+x+2$ *have no algebraical common divisor.* If, however, we put $x=6$, then

$$2x^3+x^2-x-2=460,$$

and $$3x^3-2x^2+x-2=580;$$

and the greatest common measure of 460 and 580 is 20; whereas 5 is the numerical value of $x-1$, the algebraical highest common factor. Thus the numerical values of the algebraical highest common factor and of the arithmetical greatest common measure do not in this case agree.

The reason may be explained as follows: when $x=6$, the expressions $2x^2+3x+2$ and $3x^2+x+2$ become equal to 92 and 116 respectively, and have a common arithmetical factor 4; whereas the expressions have no algebraical common factor.

It will thus often happen that the highest common factor of two expressions, and their numerical greatest common measure, when the letters have particular values, are not the same; for this reason the term *greatest common measure* is inappropriate when applied to algebraical quantities.

*EXAMPLES XVIII. b.

Find the highest common factor of the following expressions:

1. $x^3+2x^2-13x+10$, $x^3+x^2-10x+8$.

2. $x^3-5x^2-99x+40$, $x^3-6x^2-86x+35$.

3. $x^3+2x^2-8x-16$, $x^3+3x^2-8x-24$.

4. $x^3+4x^2-5x-20$, $x^3+6x^2-5x-30$.

5. x^3-x^2-5x-3, $x^3-4x^2-11x-6$.

6. $x^3+3x^2-8x-24$, x^3+3x^2-3x-9.

7. $a^3-5a^2x+7ax^2-3x^3$, $a^3-3ax^2+2x^3$.

8. x^4-2x^3-4x-7, $x^4+x^3-3x^2-x+2$.

9. $2x^3-5x^2+11x+7$, $4x^3-11x^2+25x+7$.

Find the highest common factor of the following expressions:

10. $2x^3+4x^2-7x-14, \quad 6x^3-10x^2-21x+35.$
11. $3x^4-3x^3-2x^2-x-1, \quad 9x^4-3x^3-x-1.$
12. $2x^4-2x^3+x^2+3x-6, \quad 4x^4-2x^3+3x-9.$
13. $3x^3-3ax^2+2a^2x-2a^3, \quad 3x^3+12ax^2+2a^2x+8a^3.$
14. $2x^3-9ax^2+9a^2x-7a^3, \quad 4x^3-20ax^2+20a^2x-16a^3.$
15. $10x^3+25ax^2-5a^3, \quad 4x^3+9ax^2-2a^2x-a^3.$
16. $6a^3+13a^2x-9ax^2-10x^3, \quad 9a^3+12a^2x-11ax^2-10x^3.$
17. $24x^4y+72x^3y^2-6x^2y^3-90xy^4, \quad 6x^4y^2+13x^3y^3-4x^2y^4-15xy^5.$
18. $4x^5a^2+10x^4a^3-60x^3a^4+54x^2a^5, \quad 24x^5a^3+30x^3a^5-126x^2a^6.$
19. $4x^5+14x^4+20x^3+70x^2, \quad 8x^7+28x^6-8x^5-12x^4+56x^3.$
20. $72x^3-12ax^2+72a^2x-420a^3, \quad 18x^3+42ax^2-282a^2x+270a^3.$
21. $9x^4+2x^2y^2+y^4, \quad 3x^4-8x^3y+5x^2y^2-2xy^3.$
22. $x^5-x^3-x+1, \quad x^7+x^6+x^4-1.$
23. $1+x+x^3-x^5, \quad 1-x^4-x^6+x^7.$
24. $6-8a-32a^2-18a^3, \quad 20-35a-95a^2-40a^3.$
25. $9x^2-15x^3-45x^4-12x^5, \quad 42x-49x^2-203x^3-84x^4.$
26. $3x^5-5x^3+2, \quad 2x^5-5x^2+3.$
27. $4x^5-6x^3-28x, \quad 6x^4+10x^3-17x^2-35x-14.$

*146. The statements of Art. 141 may be proved as follows.

I. If F divides A it will also divide mA.

For suppose $A=aF$, then $mA=maF$.

Thus F is a factor of mA.

II. If F divides A and B, then it will divide $mA \pm nB$.

For suppose $A=aF$, $B=bF$,

then
$$\begin{aligned} mA \pm nB &= maF \pm nbF \\ &= F(ma \pm nb). \end{aligned}$$

Thus F divides $mA \pm nB$.

*147. We may now enunciate and prove the rule for finding the highest common factor of any two compound algebraical expressions.

We suppose that any simple factors are first removed. [See Example, Art. 142.]

Let A and B be the two expressions after the simple factors have been removed. Let them be arranged in descending or ascending powers of some common letter; also let the highest power of that letter in B be not less than the highest power in A.

Divide B by A; let p be the quotient, and C the remainder. Suppose C to have a *simple* factor m. Remove this factor, and so obtain a new divisor D. Further, suppose that in order to make A divisible by D it is necessary to multiply A by a *simple* factor n. Let q be the next quotient and E the remainder. Finally, divide D by E; let r be the quotient, and suppose that there is no remainder. Then E will be the H.C.F. required.

The work will stand thus:

$$\begin{array}{l} A)\,B\,(p \\ \quad\underline{pA} \\ m)\,\overline{C} \\ \quad\;\; D)\,nA\,(q \\ \qquad\quad \underline{qD} \\ \qquad\quad E)\,D\,(r \\ \qquad\qquad \underline{rE} \end{array}$$

First, to shew that E is *a* common factor of A and B.

By examining the steps of the work, it is clear that E divides D, therefore also qD; therefore $qD+E$, therefore nA; therefore A, since n is a *simple* factor.

Again, E divides D, therefore mD, that is, C. And since E divides A and C, it also divides $pA+C$, that is, B. Hence E divides both A and B.

Secondly, to show that E is the *highest* common factor.

If not, let there be a factor X of higher dimensions than E.

Then X divides A and B, therefore $B-pA$, that is, C; therefore D (since m is a *simple* factor); therefore $nA-qD$, that is, E.

Thus X divides E; which is impossible since by hypothesis, X is of higher dimensions than E.

Therefore E is the highest common factor.

***148.** The highest common factor of three expressions A, B, C may be obtained as follows.

First determine F the highest common factor of A and B; next find G the highest common factor of F and C; then G will be the required highest common factor of A, B, C.

For F contains *every* factor which is common to A and B, and G is the highest common factor of F and C. Therefore G is the highest common factor of A, B, C.

CHAPTER XIX.

FRACTIONS.

[On first reading the subject, the student may omit the general proofs of the rules given in this chapter.

The articles and examples marked with an asterisk must be omitted by those who adopt the suggestion printed at the top of page 130.]

149. In Chapter XII. we discussed the simpler kinds of fractions, using the ordinary arithmetical rules. We here propose to give proofs of those rules, and shew that they are applicable to algebraical fractions.

DEFINITION. If a quantity x be divided into b equal parts, and a of these parts be taken, the result is called *the fraction* $\frac{\mathrm{a}}{\mathrm{b}}$ *of* x. If x be the unit, the fraction $\frac{a}{b}$ of x is called simply "the fraction $\frac{a}{b}$"; so that *the fraction* $\frac{\mathrm{a}}{\mathrm{b}}$ *represents* a *equal parts,* b *of which make up the unit.*

Note. This definition requires that a and b should be positive whole numbers. In Art. 155 we shall adopt a definition which will enable us to remove this restriction.

150. *To prove that* $\frac{\mathrm{a}}{\mathrm{b}}=\frac{\mathrm{ma}}{\mathrm{mb}}$, *where* a, b, m *are positive integers.*

By $\frac{a}{b}$ we mean a equal parts, b of which make up the unit ... (1);

by $\frac{ma}{mb}$ ....ma .... mb .... (2).

But b parts in (1) $= mb$ parts in (2);

$\therefore$ 1 part .... $= m$ ....

$\therefore$ a parts .... $= ma$ ....

that is,
$$\frac{a}{b}=\frac{ma}{mb}.$$

Conversely,
$$\frac{ma}{mb}=\frac{a}{b}.$$

Hence, *the value of a fraction is not altered if we multiply or divide the numerator and denominator by the same quantity.*

Reduction to Lowest Terms.

151. An algebraical fraction may be changed into an equivalent fraction by dividing numerator and denominator by any common factor; if this factor be the highest common factor the resulting fraction is said to be **reduced to its lowest terms.**

Example 1. Reduce to lowest terms $\dfrac{24a^3c^2x^2}{18a^3x^2-12a^2x^3}$.

$$\text{The expression} = \frac{24a^3c^2x^2}{6a^2x^2(3a-2x)} = \frac{4ac^2}{3a-2x}.$$

Example 2. Reduce to lowest terms $\dfrac{6x^2-8xy}{9xy-12y^2}$.

$$\text{The expression} = \frac{2x(3x-4y)}{3y(3x-4y)} = \frac{2x}{3y}.$$

Note. The beginner should be careful not to begin cancelling until he has expressed both numerator and denominator in the most convenient form, by resolution into factors where necessary.

EXAMPLES XIX. a.

Reduce to lowest terms:

1. $\dfrac{3a^2-6ab}{2a^2b-4ab^2}$. **2.** $\dfrac{abx+bx^2}{acx+cx^2}$. **3.** $\dfrac{ax}{a^2x^2-ax}$.

4. $\dfrac{15a^2b^2c}{100(a^3-a^2b)}$. **5.** $\dfrac{4x^2-9y^2}{4x^2+6xy}$. **6.** $\dfrac{20(x^3-y^3)}{5x^2+5xy+5y^2}$.

7. $\dfrac{x(2a^2-3ax)}{a(4a^2x-9x^3)}$. **8.** $\dfrac{x^3-2xy^2}{x^4-4x^2y^2+4y^4}$. **9.** $\dfrac{(xy-3y^2)^2}{x^3y^2-27y^5}$.

10. $\dfrac{x^2-5x}{x^2-4x-5}$. **11.** $\dfrac{3x^2+6x}{x^2+4x+4}$. **12.** $\dfrac{5a^3b+10a^2b^2}{3a^2b^2+6ab^3}$.

13. $\dfrac{x^3y+2x^2y+4xy}{x^3-8}$. **14.** $\dfrac{3a^4+9a^3b+6a^2b^2}{a^4+a^3b-2a^2b^2}$.

15. $\dfrac{x^4-14x^2-51}{x^4-2x^2-15}$. **16.** $\dfrac{x^2+xy-2y^2}{x^3-y^3}$. **17.** $\dfrac{2x^2+17x+21}{3x^2+26x+35}$.

18. $\dfrac{a^2x^2-16a^2}{ax^2+9ax+20a}$. **19.** $\dfrac{3x^2+23x+14}{3x^2+41x+26}$. **20.** $\dfrac{27a+a^4}{18a-6a^2+2a^3}$.

***152.** When the factors of the numerator and denominator cannot be determined by inspection, the fraction may be reduced to its lowest terms by dividing both numerator and denominator by the highest common factor, which may be found by the rules given in Chap. XVIII.

Example. Reduce to lowest terms $\dfrac{3x^3-13x^2+23x-21}{15x^3-38x^2-2x+21}$.

First Method. The H.C.F. of numerator and denominator is $3x-7$.

Dividing numerator and denominator by $3x-7$, we obtain as respective quotients x^2-2x+3 and $5x^2-x-3$.

Thus $\dfrac{3x^3-13x^2+23x-21}{15x^3-38x^2-2x+21}=\dfrac{(3x-7)(x^2-2x+3)}{(3x-7)(5x^2-x-3)}=\dfrac{x^2-2x+3}{5x^2-x-3}$.

This is the simplest solution for the beginner; but in this and similar cases we may often effect the reduction without actually going through the process of finding the highest common factor.

Second Method. By Art. 141, the H.C.F. of numerator and denominator must be a factor of their sum $18x^3-51x^2+21x$, that is, of $3x(3x-7)(2x-1)$. If there be a common divisor it must clearly be $3x-7$; hence arranging numerator and denominator so as to shew $3x-7$ as a factor,

$$\begin{aligned}\text{the fraction } &= \frac{x^2(3x-7)-2x(3x-7)+3(3x-7)}{5x^2(3x-7)-x(3x-7)-3(3x-7)}\\ &= \frac{(3x-7)(x^2-2x+3)}{(3x-7)(5x^2-x-3)}\\ &= \frac{x^2-2x+3}{5x^2-x-3}.\end{aligned}$$

***153.** If either numerator or denominator can readily be resolved into factors we may use the following method.

Example. Reduce to lowest terms $\dfrac{x^3+3x^2-4x}{7x^3-18x^2+6x+5}$.

The numerator $=x(x^2+3x-4)=x(x+4)(x-1)$.

Of these factors the only one which can be a common divisor is $x-1$. Hence, arranging the denominator,

$$\begin{aligned}\text{the fraction} &= \frac{x(x+4)(x-1)}{7x^2(x-1)-11x(x-1)-5(x-1)}\\ &= \frac{x(x+4)(x-1)}{(x-1)(7x^2-11x-5)}=\frac{x(x+4)}{7x^2-11x-5}.\end{aligned}$$

*EXAMPLES XIX. b.

Reduce to lowest terms:

1. $\dfrac{a^3 - a^2b - ab^2 - 2b^3}{a^3 + 3a^2b + 3ab^2 + 2b^3}$.

2. $\dfrac{x^3 - 5x^2 + 7x - 3}{x^3 - 3x + 2}$.

3. $\dfrac{a^3 + 2a^2 - 13a + 10}{a^3 + a^2 - 10a + 8}$.

4. $\dfrac{2x^3 + 5x^2y - 30xy^2 + 27y^3}{4x^3 + 5xy^2 - 21y^3}$.

5. $\dfrac{4a^3 + 12a^2b - ab^2 - 15b^3}{6a^3 + 13a^2b - 4ab^2 - 15b^3}$.

6. $\dfrac{1 + 2x^2 + x^3 + 2x^4}{1 + 3x^2 + 2x^3 + 3x^4}$.

7. $\dfrac{x^2 - 2x + 1}{3x^3 + 7x - 10}$.

8. $\dfrac{3a^3 - 3a^2b + ab^2 - b^3}{4a^2 - 5ab + b^2}$.

9. $\dfrac{4x^3 + 3ax^2 + a^3}{x^4 + ax^3 + a^3x + a^4}$.

10. $\dfrac{4x^3 - 10x^2 + 4x + 2}{3x^4 - 2x^3 - 3x + 2}$.

11. $\dfrac{16x^4 - 72x^2a^2 + 81a^4}{4x^2 + 12ax + 9a^2}$.

12. $\dfrac{6x^3 + x^2 - 5x - 2}{6x^3 + 5x^2 - 3x - 2}$.

13. $\dfrac{5x^3 + 2x^2 - 15x - 6}{7x^3 - 4x^2 - 21x + 12}$.

14. $\dfrac{4x^4 + 11x^2 + 25}{4x^4 - 9x^2 + 30x - 25}$.

15. $\dfrac{3x^3 - 27ax^2 + 78a^2x - 72a^3}{2x^3 + 10ax^2 - 4a^2x - 48a^3}$.

16. $\dfrac{ax^3 - 5a^2x^2 - 99a^3x + 40a^4}{x^4 - 6ax^3 - 86a^2x^2 + 35a^3x}$.

Multiplication and Division of Fractions.

154. Rule I. To multiply a fraction by an integer: *multiply the numerator by that integer; or, if the denominator be divisible by the integer, divide the denominator by it.*

The rule may be proved as follows:

(1) $\dfrac{a}{b}$ represents a equal parts, b of which make up the unit;

$\dfrac{ac}{b}$ represents ac equal parts, b of which make up the unit;

and the number of parts taken in the second fraction is c times the number taken in the first;

that is $$\frac{a}{b} \times c = \frac{ac}{b}.$$

(2) $$\frac{a}{bd} \times d = \frac{ad}{bd}, \text{ by the preceding case,}$$

$$= \frac{a}{b}. \qquad \text{[Art. 151.]}$$

155. By the preceding article

$$\frac{a}{b} \times b = \frac{ab}{b} = a,$$

that is, the fraction $\frac{a}{b}$ is that which must be multiplied by b in order to obtain a. But, by Art. 46, the quantity which must be multiplied by b in order to obtain a is the quotient resulting from the division of a by b; we may therefore define a fraction thus:

the fraction $\frac{\text{a}}{\text{b}}$ *is the quotient of* a *divided by* b.

156. Rule II. To divide a fraction by an integer: *divide the numerator, if it be divisible, by the integer; or if the numerator be not divisible, multiply the denominator by that integer.*

The rule may be proved as follows:

(1) $\frac{ac}{b}$ represents ac equal parts, b of which make up the unit;

$\frac{a}{b}$ represents a equal parts, b of which make up the unit.

The number of parts taken in the first fraction is c times the number taken in the second. Therefore the second fraction is the quotient of the first fraction divided by c;

that is $$\frac{ac}{b} \div c = \frac{a}{b}.$$

(2) But if the numerator be not divisible by c, we have

$$\frac{a}{b} = \frac{ac}{bc};$$

$$\therefore \quad \frac{a}{b} \div c = \frac{ac}{bc} \div c$$

$$= \frac{a}{bc}, \text{ by the preceding case.}$$

157. Rule III. To multiply together two or more fractions: *multiply together all the numerators to form a new numerator, and all the denominators to form a new denominator.*

To find the value of $$\frac{a}{b} \times \frac{c}{d}.$$

Let $$x = \frac{a}{b} \times \frac{c}{d}.$$

Multiplying each side by $b \times d$, we have

$$x \times b \times d = \frac{a}{b} \times \frac{c}{d} \times b \times d$$

$$= \frac{a}{b} \times b \times \frac{c}{d} \times d \qquad \text{[Art. 29.]}$$

$$= a \times c \qquad \text{[Art. 154.]}$$

$$\therefore\ x \times bd = ac.$$

Dividing each side by bd, we have

$$x = \frac{ac}{bd};$$

$$\therefore\ \frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}.$$

Similarly $$\frac{a}{b} \times \frac{c}{d} \times \frac{e}{f} = \frac{ace}{bdf};$$

and so for any number of fractions.

158. Rule IV. To divide one fraction by another: *invert the divisor, and proceed as in multiplication.*

Since division is the inverse of multiplication, we may define the quotient x, when $\frac{a}{b}$ is divided by $\frac{c}{d}$, to be such that

$$x \times \frac{c}{d} = \frac{a}{b}.$$

Multiplying by $\frac{d}{c}$ we have $x \times \frac{c}{d} \times \frac{d}{c} = \frac{a}{b} \times \frac{d}{c}$;

$$\therefore\ x = \frac{ad}{bc}.$$

Hence $$\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc} = \frac{a}{b} \times \frac{d}{c}, \qquad \text{[Art. 157.]}$$

which proves the rule.

Example 1. Simplify $\dfrac{2a^2+3a}{4a^3} \times \dfrac{4a^2-6a}{12a+18}$.

$$\frac{2a^2+3a}{4a^3} \times \frac{4a^2-6a}{12a+18} = \frac{a(2a+3)}{4a^3} \times \frac{2a(2a-3)}{6(2a+3)}$$

$$= \frac{2a-3}{12a},$$

by cancelling those factors which are common to both numerator and denominator.

Example 2. Simplify $\frac{6x^2-ax-2a^2}{ax-a^2}\times\frac{x-a}{9x^2-4a^2}\div\frac{2x+a}{3ax+2a^2}$.

$$\text{The expression}=\frac{6x^2-ax-2a^2}{ax-a^2}\times\frac{x-a}{9x^2-4a^2}\times\frac{3ax+2a^2}{2x+a}$$

$$=\frac{(3x-2a)(2x+a)}{a(x-a)}\times\frac{x-a}{(3x+2a)(3x-2a)}\times\frac{a(3x+2a)}{2x+a}$$

$$=1,$$

since all the factors cancel each other.

EXAMPLES XIX. c.

Simplify

1. $\frac{14x^2-7x}{12x^3+24x^2}\div\frac{2x-1}{x^2+2x}$.

2. $\frac{a^2b^2+3ab}{4a^2-1}\div\frac{ab+3}{2a+1}$.

3. $\frac{x^2-4a^2}{ax+2a^2}\times\frac{2a}{x-2a}$.

4. $\frac{a^2-121}{a^2-4}\div\frac{a+11}{a+2}$.

5. $\frac{16x^2-9a^2}{x^2-4}\times\frac{x-2}{4x-3a}$.

6. $\frac{25a^2-b^2}{9a^2x^2-4x^2}\times\frac{x(3a+2)}{5a+b}$.

7. $\frac{x^2+5x+6}{x^2-1}\times\frac{x^2-2x-3}{x^2-9}$.

8. $\frac{x^2+3x+2}{x^2+9x+20}\times\frac{x^2+7x+12}{x^2+5x+6}$.

9. $\frac{2x^2+5x+2}{x^2-4}\times\frac{x^2+4x}{2x^2+9x+4}$.

10. $\frac{2x^2+13x+15}{4x^2-9}\div\frac{2x^2+11x+5}{4x^2-1}$.

11. $\frac{x^2-14x-15}{x^2-4x-45}\div\frac{x^2-12x-45}{x^2-6x-27}$.

12. $\frac{2x^2-x-1}{2x^2+5x+2}\times\frac{4x^2+x-14}{16x^2-49}$.

13. $\frac{b^4-27b}{2b^2+5b}\times\frac{4b^2-25}{2b^2-11b+15}$.

14. $\frac{x^3-6x^2+36x}{x^2-49}\div\frac{x^4+216x}{x^2-x-42}$.

15. $\frac{64p^2q^2-z^4}{x^2-4}\times\frac{(x-2)^2}{8pq+z^2}\div\frac{x^2-4}{(x+2)^2}$.

16. $\frac{x^2-x-20}{x^2-25}\times\frac{x^2-x-2}{x^2+2x-8}\div\frac{x+1}{x^2+5x}$.

17. $\frac{x^2-18x+80}{x^2-5x-50}\times\frac{x^2-6x-7}{x^2-15x+56}\times\frac{x+5}{x-1}$.

18. $\frac{x^2-8x-9}{x^2-17x+72}\times\frac{x^2-25}{x^2-1}\div\frac{x^2+4x-5}{x^2-9x+8}$.

19. $\dfrac{4x^2+x-14}{6xy-14y} \times \dfrac{4x^2}{x^2-4} \times \dfrac{x-2}{4x-7} \div \dfrac{2x^2+4x}{3x^2-x-14}.$

20. $\dfrac{x^2+x-2}{x^2-x-20} \times \dfrac{x^2+5x+4}{x^2-x} \div \left(\dfrac{x^2+3x+2}{x^2-2x-15} \times \dfrac{x+3}{x^2}\right).$

21. $\dfrac{4x^2-16x+15}{2x^2+3x+1} \times \dfrac{x^2-6x-7}{2x^2-17x+21} \times \dfrac{4x^2-1}{4x^2-20x+25}.$

22. $\dfrac{x^4-8x}{x^2-4x-5} \times \dfrac{x^2+2x+1}{x^3-x^2-2x} \div \dfrac{x^2+2x+4}{x-5}.$

23. $\dfrac{(a+b)^2-c^2}{a^2+ab-ac} \times \dfrac{a}{(a+c)^2-b^2} \times \dfrac{(a-b)^2-c^2}{ab-b^2-bc}.$

24. $\dfrac{a^2+2ab+b^2-c^2}{a^2-b^2-c^2-2bc} \times \dfrac{a^2-2ac+c^2-b^2}{b^2-2bc+c^2-a^2}.$

25. $\dfrac{x^2-64}{x^2+24x+128} \times \dfrac{x^2+12x-64}{x^3-64} \div \dfrac{x^2-16x+64}{x^2+4x+16}.$

26. $\dfrac{(a^2+ax)^3}{a^2-x^2} \times \dfrac{(a-x)^2}{a^5+a^2x^3} \times \dfrac{a^2-ax+x^2}{a^3+2a^2x+ax^2}.$

27. $\dfrac{m^3+4m^2n+4mn^2}{3m^2n-5mn^2-2n^3} \times \dfrac{m^2-4n^2}{9m^2-3mn+n^2} \div \dfrac{(m+2n)^3}{27m^3+n^3}.$

28. $\dfrac{1+8x^3}{(2-x)^2} \times \dfrac{4x-x^3}{1-4x^2} \div \dfrac{(1-2x)^2+2x}{2-5x+2x^2}.$

29. $\dfrac{x^2(x-4)^2}{(x+4)^2-4x} \times \dfrac{64-x^3}{16-x^2} \div \dfrac{(x^2-4x)^3}{(x+4)^2}.$

30. $\dfrac{(p+q)^2-r^2}{(p+q+r)^2} \times \dfrac{p^2+pq+pr}{(p-r)^2-q^2} \div \dfrac{p^2-pq+pr}{(p-q)^2-r^2}.$

31. $\dfrac{a^4-x^4}{a^2-2ax+x^2} \div \left(\dfrac{a^2x+x^3}{a^3-x^3} \times \dfrac{a^4+a^2x^2+x^4}{a^2x-ax^2+x^3}\right).$

32. $\dfrac{a^3+8a^2b+15ab^2}{(64a^3-b^3)(a^3+b^3)} \times \dfrac{16a^4-17a^2b^2+b^4}{4a^2+21ab+5b^2} \div \dfrac{a^2+2ab-3b^2}{a^3-a^2b+ab^2}.$

CHAPTER XX.

Lowest Common Multiple.

[*The articles and examples marked with an asterisk must be omitted by those who adopt the suggestion printed at the top of page* 130.]

159. Definition. The **lowest common multiple** of two or more algebraical expressions is the expression of lowest dimensions, which is divisible by each of them without remainder.

In Chapter XI. we have explained how to write down by inspection the lowest common multiple of two or more *simple* expressions; the lowest common multiple of compound expressions which are given as the product of factors, or which can be easily resolved into factors, can be readily found by a similar method.

Example 1. The lowest common multiple of $6x^2(a-x)^2$, $8a^3(a-x)^3$ and $12ax(a-x)^5$ is $24a^3x^2(a-x)^5$.

For it consists of the product of

(1) the L.C.M. of the numerical coefficients;

(2) the lowest power of each factor which is divisible by every power of that factor occurring in the given expressions.

Example 2. Find the lowest common multiple of

$$3a^2+9ab,\quad 2a^3-18ab^2,\quad a^3+6a^2b+9ab^2.$$

$$\begin{aligned} 3a^2+9ab &= 3a(a+3b), \\ 2a^3-18ab^2 &= 2a(a+3b)(a-3b), \\ a^3+6a^2b+9ab^2 &= a(a+3b)(a+3b) \\ &= a(a+3b)^2. \end{aligned}$$

Therefore the L.C.M. is $6a(a+3b)^2(a-3b)$.

EXAMPLES XX. a.

Find the lowest common multiple of

1. $x,\ x^2+x$. **2.** $x^2,\ x^2-3x$. **3.** $3x^2,\ 4x^2+8x$.

4. $21x^3,\ 7x^2(x+1)$. **5.** $x^2-1,\ x^2+x$. **6.** $a^2+ab,\ ab+b^2$.

7. $4x^2y-y,\ 2x^2+x$. **8.** $6x^2-2x,\ 9x^2-3x$.

9. $x^2+2x,\ x^2+3x+2$. **10.** $x^2-3x+2,\ x^2-1$.

11. $x^2+4x+4,\ x^2+5x+6$. **12.** $x^2-5x+4,\ x^2-6x+8$.

13. $x^2-x-6,\ x^2+x-2,\ x^2-4x+3.$
14. $x^2+x-20,\ x^2-10x+24,\ x^2-x-30.$
15. $x^2+x-42,\ x^2-11x+30,\ x^2+2x-35.$
16. $2x^2+3x+1,\ 2x^2+5x+2,\ x^2+3x+2.$
17. $3x^2+11x+6,\ 3x^2+8x+4,\ x^2+5x+6.$
18. $5x^2+11x+2,\ 5x^2+16x+3,\ x^2+5x+6.$
19. $2x^2+3x-2,\ 2x^2+15x-8,\ x^2+10x+16.$
20. $3x^2-x-14,\ 3x^2-13x+14,\ x^2-4.$
21. $12x^2+3x-42,\ 12x^3+30x^2+12x,\ 32x^2-40x-28.$
22. $3x^4+26x^3+35x^2,\ 6x^2+38x-28,\ 27x^3+27x^2-30x.$
23. $60x^4+5x^3-5x^2,\ 60x^2y+32xy+4y,\ 40x^3y-2x^2y-2xy.$
24. $8x^2-38xy+35y^2,\ 4x^2-xy-5y^2,\ 2x^2-5xy-7y^2.$
25. $12x^2-23xy+10y^2,\ 4x^2-9xy+5y^2,\ 3x^2-5xy+2y^2.$
26. $6ax^3+7a^2x^2-3a^3x,\ 3a^2x^2+14a^3x-5a^4,\ 6x^2+39ax+45a^2.$
27. $4ax^2y^2+11axy^2-3ay^2,\ 3x^3y^3+7x^2y^3-6xy^3,\ 24ax^2-22ax+4a.$
28. $(3x-5x^2)^2,\ 6-7x-5x^2,\ 4x+4x^2+x^3.$
29. $14a^4(a^3-b^3),\ 21a^2b^2(a-b)^3,\ 6a^3b(a-b)(a^2-b^2).$
30. $m^4+m^2n^2+n^4,\ m^3n+n^4,\ (m^2-mn)^3.$
31. $(2c^2-3cd)^2,\ (4c-6d)^3,\ 8c^3-27d^3.$

***160.** When the given expressions are such that their factors cannot be determined by inspection, they must be resolved by finding the highest common factor.

Example. Find the lowest common multiple of

$$2x^4+x^3-20x^2-7x+24 \text{ and } 2x^4+3x^3-13x^2-7x+15.$$

The highest common factor is x^2+2x-3.

By division, we obtain

$$2x^4+x^3-20x^2-7x+24=(x^2+2x-3)(2x^2-3x-8).$$
$$2x^4+3x^3-13x^2-7x+15=(x^2+2x-3)(2x^2-x-5).$$

Therefore the L.C.M. is $(x^2+2x-3)(2x^2-3x-8)(2x^2-x-5)$.

***161.** We may now give the proof of the rule for finding the lowest common multiple of two compound algebraical expressions.

Let A and B be the two expressions, and F their highest common factor. Also suppose that a and b are the respective quotients when A and B are divided by F; then $A=aF$, $B=bF$. Therefore, since a and b have no common factor, the lowest common multiple of A and B is abF, by inspection.

***162.** There is an important relation between the highest common factor and the lowest common multiple of two expressions which it is desirable to notice.

Let F be the highest common factor, and X the lowest common multiple of A and B. Then, as in the preceding article,

$$A=aF, \quad B=bF,$$

and $$X=abF.$$

Therefore the product
$$\begin{aligned} AB &= aF \,.\, bF \\ &= F \,.\, abF \\ &= FX \quad \ldots\ldots\ldots\ldots\ldots\ldots(1). \end{aligned}$$

Hence *the product of two expressions is equal to the product of their highest common factor and lowest common multiple.*

Again, from (1) $\quad X=\dfrac{AB}{F}=\dfrac{A}{F}\times B=\dfrac{B}{F}\times A\,;$

hence *the lowest common multiple of two expressions may be found by dividing their product by their highest common factor; or by dividing either of them by their highest common factor, and multiplying the quotient by the other.*

***163.** The lowest common multiple of three expressions A, B, C may be obtained as follows.

First, find X the L.C.M. of A and B. Next find Y the L.C.M. of X and C; then Y will be the required L.C.M. of A, B, C.

For Y is the expression of lowest dimensions which is divisible by X and C, and X is the expression of lowest dimensions divisible by A and B. Therefore Y is the expression of lowest dimensions divisible by all three.

EXAMPLES XX. b.

1. Find the highest common factor and the lowest common multiple of x^2-5x+6, x^2-4, x^3-3x-2.

2. Find the lowest common multiple of

$$ab(x^2+1)+x(a^2+b^2) \text{ and } ab(x^2-1)+x(a^2-b^2).$$

3. Find the lowest common multiple of

$$xy-bx, \quad xy-ay, \quad y^2-3by+2b^2, \quad xy-2bx-ay+2ab,$$
$$xy-bx-ay+ab.$$

4. Find the highest common factor and the lowest common multiple of x^3+2x^2-3x, $2x^3+5x^2-3x$.

5. Find the lowest common multiple of

$$1-x,\ (1-x^2)^2,\ (1+x)^3.$$

6. Find the lowest common multiple of

$$x^2-10x+24,\ x^2-8x+12,\ x^2-6x+8.$$

7. Find the highest common factor and the lowest common multiple of $6x^3+x^2-5x-2$, $6x^3+5x^2-3x-2$.

8. Find the lowest common multiple of

$$(bc^2-abc)^2,\ b^2(ac^2-a^3),\ a^2c^2+2ac^3+c^4.$$

9. Find the lowest common multiple of

$$x^3-y^3,\ x^3y-y^4,\ y^2(x-y)^2,\ x^2+xy+y^2.$$

Also find the highest common factor of the first three expressions.

10. Find the highest common factor of

$$6x^2-13x+6,\ 2x^2+5x-12,\ 6x^2-x-12.$$

Also shew that the lowest common multiple is the product of the three quantities divided by the square of the highest common factor.

11. Find the lowest common multiple of

$$x^4+ax^3+a^3x+a^4,\ x^4+a^2x^2+a^4.$$

*12. Find the highest common factor and the lowest common multiple of $3x^3-7x^2y+5xy^2-y^3$, $x^2y+3xy^2-3x^3-y^3$,

$$3x^3+5x^2y+xy^2-y^3.$$

*13. Find the highest common factor of

$$4x^3-10x^2+4x+2,\ 3x^4-2x^3-3x+2.$$

14. Find the lowest common multiple of

$$a^2-b^2,\ a^3-b^3,\ a^3-a^2b-ab^2-2b^3.$$

15. Find the highest common factor and the lowest common multiple of $(2x^2-3a^2)y+(2a^2-3y^2)x$, $(2a^2+3y^2)x+(2x^2+3a^2)y$.

*16. Find the highest common factor and the lowest common multiple of $x^3-9x^2+26x-24$, $x^3-12x^2+47x-60$.

*17. Find the highest common factor of

$$x^3-15ax^2+48a^2x+64a^3,\ x^2-10ax+16a^2.$$

18. Find the lowest common multiple of

$$21x(xy-y^2)^2,\ 35(x^4y^2-x^2y^4),\ 15y(x^2+xy)^2.$$

CHAPTER XXI.

Addition and Subtraction of Fractions.

164. Having explained the rules for finding the lowest common multiple of any given expressions, we now proceed to shew how the addition and subtraction of fractions may be effected.

165. *To prove* $\quad \dfrac{a}{b}+\dfrac{c}{d}=\dfrac{ad+bc}{bd}.$

We have $\quad \dfrac{a}{b}=\dfrac{ad}{bd}$, and $\dfrac{c}{d}=\dfrac{bc}{bd}$. [Art. 150].

Thus in each case we divide the unit into bd equal parts, and we take first ad of these parts, and then bc of them; that is, we take $ad+bc$ of the bd parts of the unit; and this is expressed by the fraction $\dfrac{ad+bc}{bd}$.

$$\therefore \quad \frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}.$$

Similarly, $$\frac{a}{b}-\frac{c}{d}=\frac{ad-bc}{bd}.$$

166. Here the fractions have been both expressed with a common denominator bd. But if b and d have a common factor, the product bd is not the lowest common denominator, and the fraction $\dfrac{ad+bc}{bd}$ will not be in its lowest terms. To avoid working with fractions which are not in their lowest terms, some modification of the above will be necessary. In practice it will be found advisable to take the *lowest* common denominator, which is the lowest common multiple of the denominators of the given fractions.

Rule. I. To reduce fractions to their lowest common denominator: *find the L.C.M. of the given denominators, and take it for the common denominator; divide it by the denominator of the first fraction, and multiply the numerator of this fraction by the quotient so obtained; and do the same with all the other given fractions.*

Example. Express with lowest common denominator

$$\frac{5x}{2a(x-a)} \text{ and } \frac{4a}{3x(x^2-a^2)}.$$

The lowest common denominator is $6ax(x-a)(x+a)$.

We must therefore multiply the numerators by $3x(x+a)$ and $2a$ respectively.

Hence the equivalent fractions are

$$\frac{15x^2(x+a)}{6ax(x-a)(x+a)} \text{ and } \frac{8a^2}{6ax(x-a)(x+a)}.$$

167. We may now enunciate the rule for the addition or subtraction of fractions.

Rule II. To add or subtract fractions: *reduce them to the lowest common denominator; find the algebraical sum of the numerators, and retain the common denominator.*

Example 1. Find the value of $\frac{2x+a}{3a}+\frac{5x-4a}{9a}$.

The lowest common denominator is $9a$.

Therefore the expression $=\frac{3(2x+a)+5x-4a}{9a}$

$$=\frac{6x+3a+5x-4a}{9a}=\frac{11x-a}{9a}.$$

Example 2. Find the value of $\frac{x-2y}{xy}+\frac{3y-a}{ay}-\frac{3x-2a}{ax}$.

The lowest common denominator is axy.

Thus the expression $=\frac{a(x-2y)+x(3y-a)-y(3x-2a)}{axy}$

$$=\frac{ax-2ay+3xy-ax-3xy+2ay}{axy}$$

$$=0,$$

since the terms in the numerator destroy each other.

Note. To ensure accuracy the beginner is recommended to use brackets as in the first line of work above.

EXAMPLES XXI. a.

Find the value of

1. $\dfrac{x-1}{2}+\dfrac{x+3}{5}+\dfrac{x+7}{10}$.

2. $\dfrac{2x-1}{3}+\dfrac{x-5}{6}+\dfrac{x-4}{4}$.

3. $\dfrac{5x-1}{8}-\dfrac{3x-2}{7}+\dfrac{x-5}{4}$.

4. $\dfrac{2x-3}{9}-\dfrac{x+2}{6}+\dfrac{5x+8}{12}$.

5. $\dfrac{x-7}{15}+\dfrac{x-9}{25}-\dfrac{x+3}{45}$.

6. $\dfrac{2x+5}{x}-\dfrac{x+3}{2x}-\dfrac{27}{8x^2}$.

7. $\dfrac{a-b}{ab}+\dfrac{b-c}{bc}+\dfrac{c-a}{ca}$.

8. $\dfrac{a-2b}{2a}-\dfrac{a-5b}{4a}+\dfrac{a+7b}{8a}$.

9. $\dfrac{b+c}{2a}+\dfrac{c+a}{4b}-\dfrac{a-b}{3c}$.

10. $\dfrac{a-x}{x}+\dfrac{a+x}{a}-\dfrac{a^2-x^2}{2ax}$.

11. $\dfrac{x+2}{17x}-\dfrac{x-5}{34x}+\dfrac{x+2}{51x}$.

12. $\dfrac{2a^2-b^2}{a^2}-\dfrac{b^2-c^2}{b^2}-\dfrac{c^2-a^2}{c^2}$.

13. $\dfrac{x-3}{5x}+\dfrac{x^2-9}{10x^2}-\dfrac{8-x^3}{15x^3}$.

14. $\dfrac{2}{xy}-\dfrac{3y^2-x^2}{xy^3}+\dfrac{xy+y^2}{x^2y^2}$.

15. $\dfrac{2x-3y}{xy}+\dfrac{3x-2z}{xz}+\dfrac{5}{x}$.

16. $\dfrac{a^2-bc}{bc}-\dfrac{ac-b^2}{ac}-\dfrac{ab-c^2}{ab}$.

Example 3. Simplify $\dfrac{2x-3a}{x-2a}-\dfrac{2x-a}{x-a}$.

The lowest common denominator is $(x-2a)(x-a)$.

Hence, multiplying the numerators by $x-a$ and $x-2a$ respectively, we have

$$\begin{aligned}\text{the expression} &= \frac{(2x-3a)(x-a)-(2x-a)(x-2a)}{(x-2a)(x-a)}\\ &= \frac{2x^2-5ax+3a^2-(2x^2-5ax+2a^2)}{(x-2a)(x-a)}\\ &= \frac{2x^2-5ax+3a^2-2x^2+5ax-2a^2}{(x-2a)(x-a)}\\ &= \frac{a^2}{(x-2a)(x-a)}.\end{aligned}$$

Note. In finding the value of such an expression as

$$-(2x-a)(x-2a),$$

the beginner should first express the product in brackets, and then remove the brackets, as we have done. After a little practice he will be able to take both steps together.

The work will sometimes be shortened by first reducing the fractions to their lowest terms.

Example 4. Simplify $\dfrac{x^2+5xy-4y^2}{x^2-16y^2}-\dfrac{2xy}{2x^2+8xy}$.

$$\text{The expression} = \frac{x^2+5xy-4y^2}{x^2-16y^2}-\frac{y}{x+4y}$$

$$=\frac{x^2+5xy-4y^2-y(x-4y)}{x^2-16y^2}$$

$$=\frac{x^2+5xy-4y^2-xy+4y^2}{x^2-16y^2}$$

$$=\frac{x^2+4xy}{x^2-16y^2}=\frac{x}{x-4y}.$$

EXAMPLES XXI. b.

Find the value of

1. $\dfrac{1}{x+2}+\dfrac{1}{x+3}$.
2. $\dfrac{2}{x+3}-\dfrac{1}{x+4}$.
3. $\dfrac{1}{x-5}-\dfrac{1}{x-4}$.
4. $\dfrac{3}{x-6}-\dfrac{1}{x+2}$.
5. $\dfrac{a}{x+a}-\dfrac{b}{x+b}$.
6. $\dfrac{a}{x-a}+\dfrac{b}{x-b}$.
7. $\dfrac{x+3}{x+4}-\dfrac{x+1}{x+2}$.
8. $\dfrac{a+x}{a-x}-\dfrac{a-x}{a+x}$.
9. $\dfrac{x+2}{x-2}-\dfrac{x-2}{x+2}$.
10. $\dfrac{x-4}{x-2}-\dfrac{x-7}{x-5}$.
11. $\dfrac{a}{x-a}-\dfrac{a^2}{x^2-a^2}$.
12. $\dfrac{3}{x-3}+\dfrac{2x}{x^2-9}$.
13. $\dfrac{1}{2x-3y}-\dfrac{x+y}{4x^2-9y^2}$.
14. $\dfrac{x+a}{x-2a}-\dfrac{x^2+2a^2}{x^2-4a^2}$.
15. $\dfrac{4a^2+b^2}{4a^2-b^2}-\dfrac{2a-b}{2a+b}$.
16. $\dfrac{2x^2}{x^2-y^2}-\dfrac{2x^2}{x^2+xy}$.
17. $\dfrac{x^2}{x-x^3}-\dfrac{x}{1+x^2}$.
18. $\dfrac{1}{x(x-y)}+\dfrac{1}{y(x+y)}$.
19. $\dfrac{xy}{25x^2-y^2}+\dfrac{2x^2y}{10x^2y+2xy^2}$.
20. $\dfrac{y}{x(x^2-y^2)}+\dfrac{x}{y(x^2+y^2)}$.
21. $\dfrac{x^2-4a^2}{x^2-2ax}-\dfrac{x^2+2ax-8a^2}{x^2-4a^2}$.
22. $\dfrac{x^2+xy+y^2}{x+y}+\dfrac{x^2-xy+y^2}{x-y}$.
23. $\dfrac{1}{a-2x}-\dfrac{(a+2x)^2}{a^3-8x^3}$.
24. $\dfrac{a^3+b^3}{a^2-ab+b^2}-\dfrac{a^3-b^3}{a^2+ab+b^2}$.
25. $\dfrac{3}{x^2-4}+\dfrac{1}{(x-2)^2}$.
26. $\dfrac{1}{a(x^2-a^2)}-\dfrac{1}{x(x+a)^2}$.

168. Some modification of the foregoing general methods may sometimes be used with advantage. The most useful artifices are explained in the examples which follow, but no general rules can be given which will apply to all cases.

Example 1. Simplify $\dfrac{a+3}{a-4}-\dfrac{a+4}{a-3}-\dfrac{8}{a^2-16}$.

Taking the first two fractions together, we have

$$\begin{aligned}\text{the expression} &= \frac{a^2-9-(a^2-16)}{(a-4)(a-3)}-\frac{8}{a^2-16}\\ &= \frac{7}{(a-4)(a-3)}-\frac{8}{(a+4)(a-4)}\\ &= \frac{7(a+4)-8(a-3)}{(a+4)(a-4)(a-3)}\\ &= \frac{52-a}{(a+4)(a-4)(a-3)}.\end{aligned}$$

Example 2. Simplify $\dfrac{1}{2x^2+x-1}+\dfrac{1}{3x^2+4x+1}$.

$$\begin{aligned}\text{The expression} &= \frac{1}{(2x-1)(x+1)}+\frac{1}{(3x+1)(x+1)}\\ &= \frac{3x+1+2x-1}{(2x-1)(x+1)(3x+1)}\\ &= \frac{5x}{(2x-1)(x+1)(3x+1)}.\end{aligned}$$

Example 3. Simplify $\dfrac{1}{a-x}-\dfrac{1}{a+x}-\dfrac{2x}{a^2+x^2}-\dfrac{4x^3}{a^4+x^4}$.

Here it should be evident that the first two denominators give L.C.M. a^2-x^2, which readily combines with a^2+x^2 to give L.C.M. a^4-x^4, which again combines with a^4+x^4 to give L.C.M. a^8-x^8. Hence it will be convenient to proceed as follows:

$$\begin{aligned}\text{The expression} &= \frac{a+x-(a-x)}{a^2-x^2}-\dots\dots-\dots\dots\\ &= \frac{2x}{a^2-x^2}-\frac{2x}{a^2+x^2}-\dots\dots\\ &= \frac{4x^3}{a^4-x^4}-\frac{4x^3}{a^4+x^4}\\ &= \frac{8x^7}{a^8-x^8}.\end{aligned}$$

EXAMPLES XXI. c.

Find the value of

1. $\dfrac{1}{x+y}-\dfrac{1}{x-y}+\dfrac{2x}{x^2-y^2}$.

2. $\dfrac{1}{2x+y}+\dfrac{1}{2x-y}-\dfrac{3x}{4x^2-y^2}$.

3. $\dfrac{5}{1+2x}-\dfrac{3x}{1-2x}-\dfrac{4-13x}{1-4x^2}$.

4. $\dfrac{2a}{2a+3b}+\dfrac{3b}{2a-3b}-\dfrac{8b^2}{4a^2-9b^2}$.

5. $\dfrac{10}{9-a^2}-\dfrac{2}{3+a}-\dfrac{1}{3-a}$.

6. $\dfrac{5x}{6(x^2-1)}-\dfrac{1}{2(x-1)}+\dfrac{1}{3(x+1)}$.

7. $\dfrac{1}{2(a-b)}-\dfrac{1}{2(a+b)}-\dfrac{b}{a^2-b^2}$.

8. $\dfrac{2a}{2a-3}-\dfrac{5}{6a+9}-\dfrac{4(3a+2)}{3(4a^2-9)}$.

9. $\dfrac{3}{x-2}+\dfrac{2}{3x+6}+\dfrac{5x}{x^2-4}$.

10. $\dfrac{x}{x^3+y^3}-\dfrac{y}{x^3-y^3}+\dfrac{x^3y+xy^3}{x^6-y^6}$.

11. $\dfrac{1}{x^2-9x+20}+\dfrac{1}{x^2-11x+30}$.

12. $\dfrac{1}{x^2-7x+12}-\dfrac{1}{x^2-5x+6}$.

13. $\dfrac{1}{2x^2-x-1}-\dfrac{1}{2x^2+x-3}$.

14. $\dfrac{1}{2x^2-x-1}-\dfrac{3}{6x^2-x-2}$.

15. $\dfrac{4}{4-7a-2a^2}-\dfrac{3}{3-a-10a^2}$.

16. $\dfrac{5}{5+x-18x^2}-\dfrac{2}{2+5x+2x^2}$.

17. $\dfrac{1}{x+1}-\dfrac{1}{(x+1)(x+2)}+\dfrac{1}{(x+1)(x+2)(x+3)}$.

18. $\dfrac{5x}{2(x+1)(x-3)}-\dfrac{15(x-1)}{16(x-3)(x-2)}-\dfrac{9(x+3)}{16(x+1)(x-2)}$.

19. $\dfrac{a+3b}{4(a+b)(a+2b)}+\dfrac{a+2b}{(a+b)(a+3b)}-\dfrac{a+b}{4(a+2b)(a+3b)}$.

20. $\dfrac{2}{x^2-3x+2}+\dfrac{2}{x^2-x-2}-\dfrac{1}{x^2-1}$.

21. $\dfrac{x}{x^2+5x+6}+\dfrac{15}{x^2+9x+14}-\dfrac{12}{x^2+10x+21}$.

22. $\dfrac{3}{x^2-1}+\dfrac{4}{2x+1}+\dfrac{4x+2}{2x^2+3x+1}$.

23. $\dfrac{5(2x-3)}{11(6x^2+x-1)}+\dfrac{7x}{6x^2+7x-3}-\dfrac{12(3x+1)}{11(4x^2+8x+3)}$.

24. $\dfrac{x-3}{x+2}-\dfrac{x-2}{x+3}+\dfrac{1}{x-1}$.

25. $\dfrac{x-3}{x-4}-\dfrac{x+4}{x+3}-\dfrac{5}{x^2-16}$.

Find the value of

26. $\dfrac{1+2a}{1-2a}-\dfrac{1-2a}{1+2a}-\dfrac{8a}{(1-2a)^2}$.

27. $\dfrac{24x}{9-12x+4x^2}-\dfrac{3+2x}{3-2x}+\dfrac{3-2x}{3+2x}$.

28. $\dfrac{1}{3-x}-\dfrac{1}{3+x}-\dfrac{2x}{9+x^2}$.

29. $\dfrac{1}{2a+3}+\dfrac{1}{2a-3}-\dfrac{4a}{4a^2+9}$.

30. $\dfrac{1}{4(1+x)}+\dfrac{1}{4(1-x)}+\dfrac{1}{2(1+x^2)}$.

31. $\dfrac{3}{8(a-x)}+\dfrac{1}{8(a+x)}-\dfrac{a-x}{4(a^2+x^2)}$.

32. $\dfrac{2x}{4+x^2}+\dfrac{1}{2-x}-\dfrac{1}{2+x}$.

33. $\dfrac{5}{3-6x}-\dfrac{5}{3+6x}-\dfrac{x}{2+8x^2}$.

34. $\dfrac{1}{2a-8x}-\dfrac{a}{3a^2+48x^2}+\dfrac{1}{2a+8x}$.

35. $\dfrac{1}{6a^2+54}+\dfrac{1}{3a-9}-\dfrac{a}{3a^2-27}$.

36. $\dfrac{1}{8-8x}-\dfrac{1}{8+8x}+\dfrac{x}{4+4x^2}-\dfrac{x}{2+2x^4}$.

37. $\dfrac{1}{6a-18}-\dfrac{1}{6a+18}-\dfrac{1}{a^2+9}+\dfrac{18}{a^4+81}$.

38. $\dfrac{x+1}{2x^3-4x^2}+\dfrac{x-1}{2x^3+4x^2}-\dfrac{1}{x^2-4}$.

39. $\dfrac{1}{3x^2-4xy+y^2}+\dfrac{1}{x^2-4xy+3y^2}-\dfrac{3}{3x^2-10xy+3y^2}$.

40. $\dfrac{1}{x-1}+\dfrac{2}{x+1}-\dfrac{3x-2}{x^2-1}-\dfrac{1}{(x+1)^2}$.

41. $\dfrac{108-52x}{x(3-x)^2}-\dfrac{4}{3-x}-\dfrac{12}{x}+\left(\dfrac{1+x}{3-x}\right)^2$.

42. $\dfrac{(a+b)^2}{(x-a)(x+a+b)}-\dfrac{a+2b+x}{2(x-a)}+\dfrac{(a+b)x}{x^2+bx-a^2-ab}+\dfrac{1}{2}$.

43. $\dfrac{3(x^2+x-2)}{x^2-x-2}-\dfrac{3(x^2-x-2)}{x^2+x-2}-\dfrac{8x}{x^2-4}$.

169. We have thus far assumed both numerator and denominator to be positive integers, and have shewn in Art. 155 that a fraction itself is the quotient resulting from the division of the numerator by denominator. But in algebra division is a process not restricted to positive integers, and we shall extend this definition as follows:

The algebraic fraction $\dfrac{a}{b}$ *is the quotient resulting from the division of* a *by* b, *where* a *and* b *may have any values whatever.*

170. By the preceding article $\frac{-a}{-b}$ is the quotient resulting from the division of $-a$ by $-b$; and this is obtained by dividing a by b, and, by the rule of signs, prefixing $+$.

Again, $\frac{-a}{b}$ is the quotient resulting from the division of $-a$ by b; and this is obtained by dividing a by b, and, by the rule of signs, prefixing $-$.

Therefore $$\frac{-a}{b}=-\frac{a}{b}\text{..............................}(2).$$

Likewise $\frac{a}{-b}$ is the quotient resulting from the division of a by $-b$; and this is obtained by dividing a by b, and, by the rule of signs, prefixing $-$.

Therefore $$\frac{a}{-b}=-\frac{a}{b}\text{..............................}(3).$$

These results may be enunciated as follows:

1. *If the signs of* BOTH *numerator and denominator of a fraction be changed, the sign of the whole fraction will be unchanged.*

2. *If the sign of* EITHER *numerator or denominator alone be changed, the sign of the whole fraction will be changed.*

The principles here involved are so useful in certain cases of reduction of fractions that we quote them in another form, which will sometimes be found more easy of application.

1. *We may change the sign of every term in the numerator and denominator of a fraction without altering its value.*

2. *We may change the sign of a fraction by simply changing the sign of every term in* EITHER *the numerator or denominator.*

Example 1. $$\frac{b-a}{y-x}=\frac{-b+a}{-y+x}=\frac{a-b}{x-y}.$$

Example 2. $$\frac{x-x^2}{2y}=-\frac{-x+x^2}{2y}=-\frac{x^2-x}{2y}.$$

Example 3. $$\frac{3x}{4-x^2}=-\frac{3x}{-4+x^2}=-\frac{3x}{x^2-4}.$$

The intermediate step may usually be omitted.

Example 4. Simplify $\frac{a}{x+a}+\frac{2x}{x-a}+\frac{a(3x-a)}{a^2-x^2}$.

Here it is evident that the lowest common denominator of the first two fractions is x^2-a^2, therefore it will be convenient to alter the sign of the denominator in the third fraction.

$$\begin{aligned}\text{Thus the expression} &= \frac{a}{x+a}+\frac{2x}{x-a}-\frac{a(3x-a)}{x^2-a^2}\\ &= \frac{a(x-a)+2x(x+a)-a(3x-a)}{x^2-a^2}\\ &= \frac{ax-a^2+2x^2+2ax-3ax+a^2}{x^2-a^2}\\ &= \frac{2x^2}{x^2-a^2}.\end{aligned}$$

Example 5. Simplify $\frac{5}{3x-3}+\frac{3x-1}{1-x^2}+\frac{1}{2x+2}$.

$$\begin{aligned}\text{The expression} &= \frac{5}{3(x-1)}-\frac{3x-1}{x^2-1}+\frac{1}{2(x+1)}\\ &= \frac{10(x+1)-6(3x-1)+3(x-1)}{6(x^2-1)}\\ &= \frac{10x+10-18x+6+3x-3}{6(x^2-1)}\\ &= \frac{13-5x}{6(x^2-1)}.\end{aligned}$$

EXAMPLES XXI. d.

Simplify

1. $\frac{1}{4x-4}-\frac{1}{5x+5}+\frac{1}{1-x^2}$.

2. $\frac{3}{1+a}-\frac{2}{1-a}-\frac{5a}{a^2-1}$.

3. $\frac{x-2a}{x+a}+\frac{2(a^2-4ax)}{a^2-x^2}-\frac{3a}{x-a}$.

4. $\frac{x-a}{x+a}+\frac{a^2+3ax}{a^2-x^2}+\frac{x+a}{x-a}$.

5. $\frac{1}{2x+1}+\frac{1}{2x-1}+\frac{4x}{1-4x^2}$.

6. $\frac{3x}{1-x^2}-\frac{2}{x-1}-\frac{2}{x+1}$.

7. $\frac{2-5x}{x+3}-\frac{3+x}{3-x}+\frac{2x(2x-11)}{x^2-9}$.

8. $\frac{3-2x}{2x+3}-\frac{2x+3}{3-2x}+\frac{12}{4x^2-9}$.

9. $\frac{5}{2b+2}-\frac{3}{4b-4}+\frac{11}{6-6b^2}$.

10. $\frac{1}{6a+6}+\frac{1}{6-6a}-\frac{1}{3a^2-3}$.

11. $\dfrac{y^2}{x^3-y^3}+\dfrac{x^3y^2}{y^6-x^6}$.

12. $\dfrac{x^2-y^2}{xy}-\dfrac{xy-y^2}{xy-x^2}$.

13. $\dfrac{x^2+y^2}{x^2-y^2}+\dfrac{x}{x+y}+\dfrac{y}{y-x}$.

14. $\dfrac{x^2+2x+4}{x+2}-\dfrac{x^2-2x+4}{2-x}$.

15. $\dfrac{1}{2a+5b}+\dfrac{3a}{25b^2-4a^2}+\dfrac{1}{2a-5b}$.

16. $\dfrac{2b-a}{x-b}-\dfrac{3x(a-b)}{b^2-x^2}+\dfrac{b-2a}{b+x}$.

17. $\dfrac{ax^2+b}{2x-1}+\dfrac{2(bx+ax^2)}{1-4x^2}-\dfrac{ax^2-b}{2x+1}$.

18. $\dfrac{a+c}{(a-b)(x-a)}+\dfrac{b+c}{(b-a)(x-b)}$.

19. $\dfrac{a-c}{(a-b)(x-a)}-\dfrac{b-c}{(b-a)(b-x)}$.

20. $\dfrac{2a+y}{(x-a)(a-b)}+\dfrac{a+b+y}{(x-b)(b-a)}-\dfrac{x+y-a}{(x-a)(x-b)}$.

21. $\dfrac{1}{(a^2-b^2)(x^2+b^2)}+\dfrac{1}{(b^2-a^2)(x^2+a^2)}-\dfrac{1}{(x^2+a^2)(x^2+b^2)}$.

22. $\dfrac{1}{x+a}+\dfrac{4a}{x^2-a^2}+\dfrac{1}{a-x}-\dfrac{2a}{x^2+a^2}$.

23. $\dfrac{3}{x+a}-\dfrac{1}{x+3a}+\dfrac{3}{a-x}+\dfrac{1}{x-3a}$.

24. $\dfrac{1}{4a^3(a+x)}-\dfrac{1}{4a^3(x-a)}+\dfrac{1}{2a^2(a^2+x^2)}-\dfrac{a^4}{a^8-x^8}$.

25. $\dfrac{x}{x^2-y^2}-\dfrac{y}{x^2+y^2}+\dfrac{x^3+y^3}{y^4-x^4}+\dfrac{xy}{(x+y)(x^2+y^2)}$.

26. $\dfrac{b}{a(a^2-b^2)}+\dfrac{a}{b(a^2+b^2)}+\dfrac{a^4+b^4}{ab(b^4-a^4)}-\dfrac{a^6}{b^8-a^8}$.

27. $\dfrac{a^2-2ax+x^2}{2(a^2-x^2)}-\dfrac{2ax(a+x)}{(a-x)(a^2+2ax+x^2)}-\dfrac{x^2-a^2}{2(x-a)^2}$.

28. $\dfrac{2}{a+b}-\dfrac{1}{a-b}-\dfrac{3b}{b^2-a^2}+\dfrac{ab}{a^3+b^3}$.

29. $\dfrac{3}{8(1-x)}+\dfrac{1}{8(1+x)}-\dfrac{1-x}{4(1+x^2)}-\dfrac{3}{4(x^2-1)}$.

30. $\dfrac{1}{x}+\dfrac{1}{x-1}+\dfrac{1}{x+1}+\dfrac{x}{1-x^2}+\dfrac{3}{x(x^2-1)}$.

31. $\dfrac{a^2+ac}{a^2c-c^3}-\dfrac{a^2-c^2}{a^2c+2ac^2+c^3}+\dfrac{2c}{c^2-a^2}-\dfrac{3}{a+c}$.

32. $\dfrac{4a+6b}{a+b}+\dfrac{6a-4b}{a-b}+\dfrac{4a^2+6b^2}{b^2-a^2}+\dfrac{4b^2-6a^2}{a^2+b^2}-\dfrac{20b^4}{b^4-a^4}$

***171.** Consider the expression

$$\frac{1}{(a-b)(a-c)}+\frac{1}{(b-c)(b-a)}+\frac{1}{(c-a)(c-b)}.$$

Here in finding the L.C.M. of the denominators it must be observed that there are not *six* different compound factors to be considered ; for three of them differ from the other three only in sign.

Thus
$$(a-c)=-(c-a),$$
$$(b-a)=-(a-b),$$
$$(c-b)=-(b-c).$$

Hence, replacing the second factor in each denominator by its equivalent, we may write the expression in the form

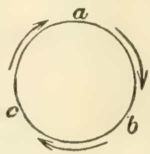
$$-\frac{1}{(a-b)(c-a)}-\frac{1}{(b-c)(a-b)}-\frac{1}{(c-a)(b-c)} \quad \text{............(1).}$$

Now the L.C.M. is $(b-c)(c-a)(a-b)$;

and the expression $=\dfrac{-(b-c)-(c-a)-(a-b)}{(b-c)(c-a)(a-b)}$

$$=\frac{-b+c-c+a-a+b}{(b-c)(c-a)(a-b)}$$
$$=0.$$

***172.** There is a peculiarity in the arrangement of this example which it is desirable to notice. In the expression (1) the letters occur in what is known as **Cyclic Order**; that is, b follows a, a follows c, c follows b. Thus if a, b, c are arranged round the circumference of a circle, as in the annexed diagram, if we start from any letter and move round in the direction of the arrows, the other letters follow in cyclic order, namely, abc, bca, cab.



The observance of this principle is especially important in a large class of examples in which the differences of three letters are involved. Thus we are observing cyclic order when we write $b-c$, $c-a$, $a-b$; whereas we are violating cyclic order by the use of arrangements such as $b-c$, $a-c$, $a-b$, or $a-c$, $b-a$, $b-c$. It will always be found that the work is rendered shorter and easier by following cyclic order from the beginning, and adhering to it throughout the question.

In the present chapter we shall confine our attention to a few of the simpler cases, resuming the subject in Chapter XXIX.

*EXAMPLES XXI. e.

Find the value of

1. $\frac{a}{(a-b)(a-c)}+\frac{b}{(b-c)(b-a)}+\frac{c}{(c-a)(c-b)}$.

2. $\frac{b}{(a-b)(a-c)}+\frac{c}{(b-c)(b-a)}+\frac{a}{(c-a)(c-b)}$.

3. $\frac{z}{(x-y)(x-z)}+\frac{x}{(y-z)(y-x)}+\frac{y}{(z-x)(z-y)}$.

4. $\frac{y+z}{(x-y)(x-z)}+\frac{z+x}{(y-z)(y-x)}+\frac{x+y}{(z-x)(z-y)}$.

5. $\frac{b-c}{(a-b)(a-c)}+\frac{c-a}{(b-c)(b-a)}+\frac{a-b}{(c-a)(c-b)}$.

6. $\frac{x^2yz}{(x-y)(x-z)}+\frac{y^2zx}{(y-z)(y-x)}+\frac{z^2xy}{(z-x)(z-y)}$.

7. $\frac{1+a}{(a-b)(a-c)}+\frac{1+b}{(b-c)(b-a)}+\frac{1+c}{(c-a)(c-b)}$.

8. $\frac{p-a}{(p-q)(p-r)}+\frac{q-a}{(q-r)(q-p)}+\frac{r-a}{(r-p)(r-q)}$.

9. $\frac{p+q-r}{(p-q)(p-r)}+\frac{q+r-p}{(q-r)(q-p)}+\frac{r+p-q}{(r-p)(r-q)}$.

10. $\frac{a^2}{(a^2-b^2)(a^2-c^2)}+\frac{b^2}{(b^2-c^2)(b^2-a^2)}+\frac{c^2}{(c^2-a^2)(c^2-b^2)}$.

11. $\frac{x+y}{(p-q)(p-r)}+\frac{x+y}{(q-r)(q-p)}+\frac{x+y}{(r-p)(r-q)}$.

12. $\frac{q+r}{(x-y)(x-z)}+\frac{r+p}{(y-z)(y-x)}+\frac{p+q}{(z-x)(z-y)}$.

CHAPTER XXII.

Miscellaneous Fractions.

[*Examples marked with an asterisk may be taken at a later stage.*]

173. We now propose to consider some miscellaneous questions involving fractions of a more complicated kind than those already discussed.

In the previous chapters on Fractions, the numerator and denominator have been regarded as integers; but cases frequently occur in which the numerator or denominator of a fraction is itself fractional.

174. Definition. A fraction of which the numerator or denominator is itself a fraction is called a **Complex Fraction.**

Thus $\dfrac{a}{\dfrac{b}{c}}$, $\dfrac{\dfrac{a}{b}}{x}$, $\dfrac{\dfrac{a}{b}}{\dfrac{c}{d}}$ are Complex Fractions.

In the last of these types, the outside quantities, a and d, are sometimes referred to as the *extremes*, while the two middle quantities, b and c, are called the *means.*

175. Instead of using the horizontal line to separate numerator and denominator, it is sometimes convenient to write complex fractions in the forms

$$a\Big/\frac{b}{c},\quad \frac{a}{b}\Big/x,\quad \frac{a}{b}\Big/\frac{c}{d}.$$

176. By definition (Art. 169) $\dfrac{\dfrac{a}{b}}{\dfrac{c}{d}}$ is the quotient resulting from the division of $\dfrac{a}{b}$ by $\dfrac{c}{d}$; and this by Art. 158 is $\dfrac{ad}{bc}$;

$$\therefore\ \frac{\dfrac{a}{b}}{\dfrac{c}{d}}=\frac{ad}{bc}.$$

Simplification of Complex Fractions.

177. From the preceding article we deduce an easy method of writing down the simplified form of a complex fraction.

Multiply the extremes for a new numerator, and the means for a new denominator.

Example.
$$\frac{\dfrac{a+x}{b}}{\dfrac{a^2-x^2}{ab}}=\frac{ab(a+x)}{b(a^2-x^2)}=\frac{a}{a-x},$$

by cancelling common factors in numerator and denominator.

178. The student should especially notice the following cases, and should be able to write down the results readily.

$$\frac{1}{\dfrac{a}{b}}=1\div\frac{a}{b}=1\times\frac{b}{a}=\frac{b}{a},$$

$$\frac{a}{\dfrac{1}{b}}=a\div\frac{1}{b}=a\times b=ab,$$

$$\frac{\dfrac{1}{a}}{\dfrac{1}{b}}=\frac{1}{a}\div\frac{1}{b}=\frac{1}{a}\times\frac{b}{1}=\frac{b}{a}.$$

179. The following examples illustrate the simplification of complex fractions.

Example 1.
$$\frac{\dfrac{a}{b}+\dfrac{c}{d}}{\dfrac{a}{b}-\dfrac{c}{d}}=\left(\frac{a}{b}+\frac{c}{d}\right)\div\left(\frac{a}{b}-\frac{c}{d}\right)=\frac{ad+bc}{bd}\div\frac{ad-bc}{bd}$$
$$=\frac{ad+bc}{bd}\times\frac{bd}{ad-bc}=\frac{ad+bc}{ad-bc}.$$

Or more simply thus :

Multiply the fractions above and below by bd which is the L.C.M. of their denominators.

Then the fraction becomes $\dfrac{ad+bc}{ad-bc}$, as before.

Example 2. Simplify $\dfrac{x = \dfrac{x^2}{x}}{x - \dfrac{a^4}{x^3}}$.

Here by multiplying above and below by x^3, we have

$$\text{the fraction} = \frac{x^4 + a^2x^2}{x^4 - a^4} = \frac{x^2(x^2 + a^2)}{x^4 - a^4}$$

$$= \frac{x^2}{x^2 - a^2}.$$

Example 3. Simplify $\dfrac{\dfrac{3}{a} + \dfrac{a}{3} - 2}{\dfrac{a}{6} + \dfrac{1}{2} - \dfrac{3}{a}}$.

$$\text{Here the expression} = \frac{18 + 2a^2 - 12a}{a^2 + 3a - 18}$$

$$= \frac{2(a^2 - 6a + 9)}{(a + 6)(a - 3)} = \frac{2(a - 3)}{a + 6}.$$

Example 4. Simplify $\dfrac{\dfrac{a^2 + b^2}{a^2 - b^2} - \dfrac{a^2 - b^2}{a^2 + b^2}}{\dfrac{a + b}{a - b} - \dfrac{a - b}{a + b}}$.

$$\text{The numerator} = \frac{(a^2 + b^2)^2 - (a^2 - b^2)^2}{(a^2 + b^2)(a^2 - b^2)}$$

$$= \frac{4a^2b^2}{(a^2 + b^2)(a^2 - b^2)}.$$

$$\text{Similarly the denominator} = \frac{4ab}{(a + b)(a - b)}.$$

$$\text{Hence the fraction} = \frac{4a^2b^2}{(a^2 + b^2)(a^2 - b^2)} \div \frac{4ab}{(a + b)(a - b)}$$

$$= \frac{4a^2b^2}{(a^2 + b^2)(a^2 - b^2)} \times \frac{(a + b)(a - b)}{4ab}$$

$$= \frac{ab}{a^2 + b^2}.$$

Note. To ensure accuracy and neatness, when the numerator and denominator are somewhat complicated, the beginner is advised to simplify each separately as in the above example.

180. In the case of fractions like the following, called **Continued Fractions,** we begin from the lowest fraction and simplify step by step.

Example. Simplify $\dfrac{9x^2-64}{x-1-\dfrac{1}{1-\dfrac{x}{4+x}}}$.

The expression $=\dfrac{9x^2-64}{x-1-\dfrac{1}{\dfrac{4+x-x}{4+x}}}=\dfrac{9x^2-64}{x-1-\dfrac{4+x}{4}}$

$$=\frac{9x^2-64}{\dfrac{4x-4-(4+x)}{4}}=\frac{9x^2-64}{\dfrac{3x-8}{4}}$$

$$=\frac{4(9x^2-64)}{3x-8}=4(3x+8).$$

EXAMPLES XXII. a.

Find the value of

1. $\dfrac{\dfrac{m}{n}-\dfrac{l}{m}}{\dfrac{a}{m}-\dfrac{b}{n}}$.
2. $\dfrac{\dfrac{1}{x}+\dfrac{1}{y}}{\dfrac{1}{x}-\dfrac{1}{y}}$.
3. $\dfrac{a+\dfrac{b}{d}}{x-\dfrac{y}{d}}$.
4. $\dfrac{1+\dfrac{c}{x}}{\dfrac{b}{x}-1}$.
5. $\dfrac{2+\dfrac{3a}{4b}}{a+\dfrac{8b}{3}}$.
6. $\dfrac{3a+\dfrac{7b}{8c}}{3c+\dfrac{7b}{8a}}$.
7. $\dfrac{1-\dfrac{y^2}{x^2}}{1+\dfrac{y^2}{x^2}}$.
8. $\dfrac{1}{a+\dfrac{b}{c}}$.
9. $\dfrac{a}{b+\dfrac{c}{d}}$.
10. $\dfrac{x}{x-\dfrac{m}{n}}$.
11. $\dfrac{\dfrac{a}{b}+\dfrac{c}{d}}{\dfrac{m}{n}+\dfrac{k}{p}}$.
12. $\dfrac{x-\dfrac{1}{x}}{1+\dfrac{1}{x}}$.
13. $\dfrac{x+5+\dfrac{6}{x}}{1+\dfrac{6}{x}+\dfrac{8}{x^2}}$.
14. $\dfrac{\dfrac{1}{x}-\dfrac{2}{x^2}-\dfrac{3}{x^3}}{\dfrac{9}{x}-x}$.
15. $\dfrac{2x^2-x-6}{\dfrac{4}{x^2}-1}$.

Find the value of

16. $\dfrac{2}{1-x^2} \div \left(\dfrac{1}{1-x} - \dfrac{1}{1+x}\right).$

17. $\left(\dfrac{a^3-b^3}{a-b} - \dfrac{a^3+b^3}{a+b}\right) \div \dfrac{4ab}{a^2-b^2}.$

18. $\left(\dfrac{a^2-ax+x^2}{a-x} - \dfrac{a^2+ax+x^2}{a+x}\right) \div \dfrac{x^3}{a^2-x^2}.$

19. $\left(y+\dfrac{xy}{y-x}\right)\left(y-\dfrac{xy}{x+y}\right) \times \dfrac{y^2-x^2}{y^2+x^2}.$

20. $\left(\dfrac{x}{1+x}+\dfrac{1-x}{x}\right) \div \left(\dfrac{x}{1+x}-\dfrac{1-x}{x}\right).$

21. $\dfrac{\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}}{1-\dfrac{a^2+b^2}{(a+b)^2}}.$

22. $\dfrac{\dfrac{a}{x^2}+\dfrac{x}{a^2}}{\dfrac{1}{a^2}-\dfrac{1}{ax}+\dfrac{1}{x^2}}.$

23. $\dfrac{\dfrac{1}{3x-2}-\dfrac{1}{3x+2}}{\dfrac{1}{9-\dfrac{4}{x^2}}}.$

24. $1+\dfrac{x}{1+x+\dfrac{2x^2}{1-x}}.$

25. $\dfrac{1}{a-\dfrac{a^2-1}{a+\dfrac{1}{a-1}}}.$

26. $\dfrac{1}{4x+\dfrac{4x}{1+\dfrac{2(x+y)}{6-x}}}.$

27. $\dfrac{a}{x+\dfrac{m}{y+\dfrac{n}{z}}}.$

28. $\dfrac{1}{1-\dfrac{1+x}{x-\dfrac{1}{x}}}.$

*29. $\dfrac{x-2}{x-2-\dfrac{x}{x-\dfrac{x-1}{x-2}}}.$

*30. $\dfrac{1}{x-\dfrac{1}{x+\dfrac{1}{x}}}-\dfrac{1}{x+\dfrac{1}{x-\dfrac{1}{x}}}.$

*31. $\dfrac{\dfrac{a-b}{1+ab}+\dfrac{b-c}{1+bc}}{1-\dfrac{(a-b)(b-c)}{(1+ab)(1+bc)}}.$

*32. $\dfrac{1+\dfrac{a-b}{a+b}}{1-\dfrac{a-b}{a+b}} \div \dfrac{1+\dfrac{a^2-b^2}{a^2+b^2}}{1-\dfrac{a^2-b^2}{a^2+b^2}}.$

*33. $\dfrac{a-x}{a^2-ax-\dfrac{(a-x)^2}{1-\dfrac{a}{x}}}$.

*34. $\dfrac{2x^2+2x}{\dfrac{x}{\dfrac{x}{x-2}-1}-\dfrac{3}{2}}-\dfrac{3(x-1)}{\dfrac{x}{\dfrac{x}{x-4}-1}+\dfrac{3}{4}}$.

*35. $\dfrac{2-4x}{4x-2-\dfrac{4x}{1+\dfrac{\dfrac{1}{2x-1}}{1+\dfrac{1}{4x-1}}}}$.

*36. $\dfrac{x^2}{1-\dfrac{1}{x^2+\dfrac{\dfrac{1}{x}}{x+\dfrac{1}{x}}}}+\dfrac{x^2-2}{1-\dfrac{1}{x^2-\dfrac{\dfrac{1}{x}}{x-\dfrac{1}{x}}}}$.

181. Sometimes it is convenient to express a single fraction as a group of fractions.

Example.
$$\frac{5x^2y-10xy^2+15y^3}{10x^2y^2}=\frac{5x^2y}{10x^2y^2}-\frac{10xy^2}{10x^2y^2}+\frac{15y^3}{10x^2y^2}$$
$$=\frac{1}{2y}-\frac{1}{x}+\frac{3y}{2x^2}.$$

182. Since a fraction represents the quotient of the numerator by the denominator, we may often express a fraction in an equivalent form, partly integral and partly fractional.

Example 1. $\dfrac{x+7}{x+2}=\dfrac{(x+2)+5}{x+2}=1+\dfrac{5}{x+2}$.

Example 2. $\dfrac{3x-2}{x+5}=\dfrac{3(x+5)-15-2}{x+5}=\dfrac{3(x+5)-17}{x+5}=3-\dfrac{17}{x+5}$.

In some cases actual division may be advisable.

Example 3. Shew that $\dfrac{2x^2-7x-1}{x-3}=2x-1-\dfrac{4}{x-3}$.

By division,
$$\begin{array}{r} x-3\,)\,2x^2-7x-1\,(\,2x-1 \\ \underline{2x^2-6x} \\ -x-1 \\ \underline{-x+3} \\ -4 \end{array}$$

Thus the quotient is $2x-1$, and the remainder -4.

Therefore
$$\frac{2x^2-7x-1}{x-3}=2x-1-\frac{4}{x-3}.$$

183. If the numerator be of lower dimensions than the denominator, we may still perform the division, and express the result in a form which is partly integral and partly fractional.

Example. Prove that $\dfrac{2x}{1+3x^2}=2x-6x^3+18x^5-\dfrac{54x^7}{1+3x^2}$.

By division

$$\begin{array}{r|l}
1+3x^2 & 2x \qquad (2x-6x^3+18x^5 \\
 & \underline{2x+6x^3} \\
 & \quad -6x^3 \\
 & \quad \underline{-6x^3-18x^5} \\
 & \qquad 18x^5 \\
 & \qquad \underline{18x^5+54x^7} \\
 & \qquad\qquad -54x^7
\end{array}$$

whence the result follows.

Here the division may be carried on to any number of terms in the quotient, and we can stop at any term we please by taking for our remainder the fraction whose numerator is the remainder last found, and whose denominator is the divisor.

Thus, if we carried on the quotient to four terms, we should have

$$\frac{2x}{1+3x^2}=2x-6x^3+18x^5-54x^7+\frac{162x^9}{1+3x^2}.$$

The terms in the quotient may be fractional; thus if x^2 is divided by x^3-a^3, the first four terms of the quotient are $\dfrac{1}{x}+\dfrac{a^3}{x^4}+\dfrac{a^6}{x^7}+\dfrac{a^9}{x^{10}}$, and the remainder is $\dfrac{a^{12}}{x^{10}}$.

184. Miscellaneous examples in multiplication and division occur which can be dealt with by the preceding rules for the reduction of fractions.

Example. Multiply $x+2a-\dfrac{a^2}{2x+3a}$ by $2x-a-\dfrac{2a^2}{x+a}$.

$$\begin{aligned}
\text{The product} &= \left(x+2a-\frac{a^2}{2x+3a}\right)\times\left(2x-a-\frac{2a^2}{x+a}\right)\\
&= \frac{2x^2+7ax+6a^2-a^2}{2x+3a}\times\frac{2x^2+ax-a^2-2a^2}{x+a}\\
&= \frac{2x^2+7ax+5a^2}{2x+3a}\times\frac{2x^2+ax-3a^2}{x+a}\\
&= \frac{(2x+5a)(x+a)}{2x+3a}\times\frac{(2x+3a)(x-a)}{x+a}\\
&= (2x+5a)(x-a).
\end{aligned}$$

EXAMPLES XXII. b.

Express each of the following fractions as a group of simple fractions in lowest terms:

1. $\dfrac{3x^2y + xy^2 - y^3}{9xy}$.
2. $\dfrac{3a^3x - 4a^2x^2 + 6ax^3}{12ax}$.
3. $\dfrac{a^3 - 3a^2b + 3ab^2 + b^3}{2ab}$.
4. $\dfrac{a+b+c}{abc}$.
5. $\dfrac{bc+ca+ab}{abc}$.
6. $\dfrac{a^3bc - 3ab^3c + 2abc}{6abc}$.

Perform the following divisions, giving the remainder after four terms in the quotient:

7. $x \div (1+x)$.
8. $a \div (a-b)$.
9. $(1+x) \div (1-x)$.
10. $1 \div (1 - x + x^2)$.
11. $x^2 \div (x+3)$.
12. $1 \div (1-x)^2$.
13. Shew that $\dfrac{a^3 - b^3}{(a-b)^2} = a + 2b + \dfrac{3b^2}{a-b}$.
14. Shew that $x^2 - xy + y^2 - \dfrac{2y^3}{x+y} = \dfrac{x^3 - y^3}{x+y}$.
15. Shew that $\dfrac{60x^3 - 17x^2 - 4x + 1}{5x^2 + 9x - 2} = 12x - 25 + \dfrac{49}{x+2}$.
16. Shew that $1 + \dfrac{a^2 + b^2 - c^2}{2ab} = \dfrac{(a+b+c)(a+b-c)}{2ab}$.
17. Divide $x + \dfrac{16x - 27}{x^2 - 16}$ by $x - 1 + \dfrac{13}{x+4}$.
18. Multiply $a^2 - 2ax + 4x^2 - \dfrac{16x^3}{a+2x}$ by $3 - \dfrac{6x(a+4x)}{a^2 + 2ax + 4x^2}$.
19. Divide $b^2 + 3b - 2 - \dfrac{12}{b-3}$ by $3b + 6 - \dfrac{2b^2}{b-3}$.
20. Divide $a^2 + 9b^2 + \dfrac{65b^4}{a^2 - 9b^2}$ by $a + 3b + \dfrac{13b^2}{a-3b}$.
21. Multiply $4x^2 + 14x + \dfrac{98x - 27}{2x - 7}$ by $\dfrac{1}{6} - \dfrac{3x + 29}{12x^2 + 18x + 27}$.

185. The following exercise contains miscellaneous examples which illustrate most of the processes connected with fractions.

*EXAMPLES XXII. c.

Simplify the following fractions :

1. $\dfrac{4a(a^2-x^2)}{3b(c^2-x^2)} \div \left[\dfrac{a^2-ax}{bc+bx} \times \dfrac{a^2+2ax+x^2}{c^2-2cx+x^2}\right].$

2. $\dfrac{x(x+a)(x+2a)}{3a} - \dfrac{x(x+a)(2x+a)}{6a}.$

3. $\dfrac{1}{b}\left(\dfrac{1}{a-b} - \dfrac{1}{a+2b}\right) - \dfrac{2}{a^2+ab-2b^2}.$

4. $\left(\dfrac{x+y}{x-y}\right)^2 - \left(\dfrac{x-y}{x+y}\right)^2.$

5. $\dfrac{2}{x-1} + \dfrac{2}{x+1} - \dfrac{4x}{x^2-x+1}.$

6. $\left(\dfrac{x^2}{1-x^4} + \dfrac{2x^4}{1-x^8}\right) \div \left(\dfrac{x^2+1}{x}\right)^2.$

7. $\dfrac{1}{x} - \dfrac{1}{(x+1)^2} - \dfrac{2}{x+1} + \dfrac{x}{1+x+x^2}.$

8. $\dfrac{1+x^3}{1+2x+2x^2+x^3}.$

9. $\dfrac{2x^3-9x^2+27}{3x^3-81x+162}.$

10. $\dfrac{a}{b} - \dfrac{(a^2-b^2)x}{b^2} + \dfrac{a(a^2-b^2)x^2}{b^2(b+ax)}.$

11. $\left\{\dfrac{x^4-a^4}{x^2-2ax+a^2} \div \dfrac{x^2+ax}{x-a}\right\} \times \dfrac{x^5-a^2x^3}{x^3+a^3} \div \left(\dfrac{x}{a} - \dfrac{a}{x}\right).$

12. $\dfrac{a^2-x^2}{a^2+ax+x^2} \div \dfrac{\left(1-\dfrac{x}{a}\right)^3\left(1+\dfrac{x}{a}\right)^3}{a^3-x^3}.$

13. $\dfrac{x^4-2x^2+1}{3x^5-10x^3+15x-8}.$

14. $\dfrac{a^3+a(1+a)y+y^2}{a^4-y^2}.$

15. $\dfrac{1}{a} + \dfrac{2}{a+1} + \dfrac{3}{a+2} - \dfrac{\dfrac{4}{a}}{1+\dfrac{1}{a}}.$

16. $\dfrac{x+3}{2x^2+9x+9} + \dfrac{1}{2} \cdot \dfrac{1}{2x-3} - \dfrac{1}{x-\dfrac{9}{4x}}.$

17. $\dfrac{2}{x^3+x^2+x+1}-\dfrac{2}{x^3-x^2+x-1}$.

18. $\dfrac{1-a^2}{(1+ax)^2-(a+x)^2}\div\dfrac{1}{2}\left(\dfrac{1}{1-x}+\dfrac{1}{1+x}\right)$.

19. $\dfrac{2x^3-x^2-2x+1}{x^3-3x+2}$.

20. $\dfrac{x^2-6x+8}{4x^3-21x^2+15x+20}$.

21. $\dfrac{2a}{(x-2a)^2}-\dfrac{x-a}{x^2-5ax+6a^2}+\dfrac{2}{x-3a}$.

22. $\dfrac{1}{2}\left(\dfrac{a^2+x^2}{a^2-x^2}\right)-\dfrac{1}{2}\cdot\dfrac{a+x}{a-x}-\left(\dfrac{a}{a+x}\right)^2$.

23. $\dfrac{x}{2}\left(\dfrac{1}{x-y}-\dfrac{1}{x+y}\right)\times\dfrac{x^2-y^2}{x^2y+xy^2}\div\dfrac{1}{x+y}$.

24. $\dfrac{1}{x+y}\div\left[\dfrac{y}{2}\left(\dfrac{1}{x+y}+\dfrac{1}{x-y}\right)\times\dfrac{x^2-y^2}{x^2y+xy^2}\right]$.

25. $\left(3x-5-\dfrac{2}{x}\right)\left(3x+5-\dfrac{2}{x}\right)\div\left(x-\dfrac{4}{x}\right)$.

26. $\left\{\dfrac{2}{x}-\dfrac{1}{a+x}+\dfrac{1}{a-x}\right\}\div\left(\dfrac{a+x}{a-x}-\dfrac{a-x}{a+x}\right)$.

27. $\dfrac{1}{2x-1}-\dfrac{2x-\dfrac{1}{2x}}{4x^2-1}$.

28. $\left(b+\dfrac{ab}{b-a}\right)\left(b-\dfrac{ab}{a+b}\right)\left(\dfrac{b^2-a^2}{b^2+a^2}\right)$.

29. $\left\{\dfrac{b+\dfrac{a-b}{1+ab}}{1-\dfrac{(a-b)b}{1+ab}}-\dfrac{a-\dfrac{a-b}{1-ab}}{1-\dfrac{a(a-b)}{1-ab}}\right\}\div\left(\dfrac{a}{b}-\dfrac{b}{a}\right)$.

30. $\dfrac{\dfrac{x^2+y^2}{y}-x}{\dfrac{1}{y}-\dfrac{1}{x}}\times\dfrac{x^2-y^2}{x^3+y^3}$.

31. $\dfrac{2\left(x^2-\dfrac{1}{4}\right)}{2x+1}+\dfrac{1}{2}$.

32. $\dfrac{a+\dfrac{b-a}{1+ab}}{1-\dfrac{a(b-a)}{1+ab}}\times\dfrac{\dfrac{x+y}{1-xy}-y}{1+\dfrac{y(x+y)}{1-xy}}$.

33. $\dfrac{\dfrac{a+b}{a-b}+\dfrac{a-b}{a+b}}{\dfrac{a-b}{a+b}-\dfrac{a+b}{a-b}}\times\dfrac{ab^3-a^3b}{a^2+b^2}$.

Simplify the following fractions:

34. $\dfrac{(1-x^2)(1-x^3)}{x(1+x)(1-x)^2}-\dfrac{x^3+\dfrac{1}{x^3}}{x^2+\dfrac{1}{x^2}-1}$.

35. $\left\{x^3-\dfrac{1}{x^3}-3\left(x-\dfrac{1}{x}\right)\right\}\div\left(x-\dfrac{1}{x}\right)$.

36. $\dfrac{1+\dfrac{1}{m}}{\dfrac{1}{m}}\times\dfrac{\dfrac{1}{m}}{m^2+\dfrac{1}{m}}\div\dfrac{\dfrac{1}{m}}{m-1+\dfrac{1}{m}}$.

37. $\dfrac{\dfrac{x}{y}+\dfrac{y}{x}-1}{\dfrac{x^2}{y^2}+\dfrac{x}{y}+1}\times\dfrac{1+\dfrac{y}{x}}{x-y}\div\dfrac{1+\dfrac{y^3}{x^3}}{\dfrac{x^2}{y}-\dfrac{y^2}{x}}$.

38. $\dfrac{\left(\dfrac{3x+x^3}{1+3x^2}\right)^2-1}{\dfrac{3x^2-1}{x^3-3x}+1}\div\dfrac{\dfrac{9}{x^2}-\dfrac{33-x^2}{3x^2+1}}{\dfrac{3}{x^2}-\dfrac{2(x^2+3)}{(x^3-x)^2}}$.

39. $\dfrac{1}{a^2-2}-\dfrac{2}{a^2-1}+\dfrac{2}{a^2+1}-\dfrac{1}{a^2+2}$.

40. $\dfrac{1}{6m-2n}+\dfrac{1}{3m+2n}-\dfrac{3}{6m+2n}$.

41. $\dfrac{3}{4(1-x)^2}+\dfrac{3}{8(1-x)}+\dfrac{1}{8(1+x)}+\dfrac{x-1}{4(1+x^2)}$.

42. $\dfrac{4}{9(x-2)}+\dfrac{5}{9(x+1)}-\dfrac{1}{3(x+1)^2}-\dfrac{1}{x+2+\dfrac{1}{x}}$.

43. $\left(\dfrac{x^2}{y}+\dfrac{y^2}{x}\right)\left(\dfrac{1}{y^2-x^2}\right)-\dfrac{y}{x^2+xy}+\dfrac{x}{xy-y^2}$.

44. $\dfrac{x^2-(y-z)^2}{(x+z)^2-y^2}+\dfrac{y^2-(z-x)^2}{(x+y)^2-z^2}+\dfrac{z^2-(x-y)^2}{(y+z)^2-x^2}$.

45. $\dfrac{x^2-(y-2z)^2}{(2z+x)^2-y^2}+\dfrac{y^2-(2z-x)^2}{(x+y)^2-4z^2}+\dfrac{4z^2-(x-y)^2}{(y+2z)^2-x^2}$.

46. $\dfrac{(x-y)(y-z)+(y-z)(z-x)+(z-x)(x-y)}{x(z-x)+y(x-y)+z(y-z)}$.

47. $\dfrac{a-b-c}{(a-b)(a-c)}+\dfrac{b-c-a}{(b-c)(b-a)}+\dfrac{c-a-b}{(c-a)(c-b)}.$

48. $\dfrac{c+a}{(a-b)(a-c)}+\dfrac{a+b}{(b-c)(b-a)}+\dfrac{b+c}{(c-a)(c-b)}.$

49. $\dfrac{x^2-(2y-3z)^2}{(3z+x)^2-4y^2}+\dfrac{4y^2-(3z-x)^2}{(x+2y)^2-9z^2}+\dfrac{9z^2-(x-2y)^2}{(2y+3z)^2-x^2}.$

50. $\dfrac{9y^2-(4z-2x)^2}{(2x+3y)^2-16z^2}+\dfrac{16z^2-(2x-3y)^2}{(3y+4z)^2-4x^2}+\dfrac{4x^2-(3y-4z)^2}{(4z+2x)^2-9y^2}.$

51. $\dfrac{\dfrac{1}{x}-\dfrac{x+a}{x^2+a^2}}{\dfrac{1}{a}-\dfrac{a+x}{a^2+x^2}}+\dfrac{\dfrac{1}{x}-\dfrac{x-a}{x^2+a^2}}{\dfrac{1}{a}-\dfrac{a-x}{a^2+x^2}}.$

52. $\dfrac{(x+a)(x+b)-(y+a)(y+b)}{x-y}-\dfrac{(x-a)(y-b)-(x-b)(y-a)}{a-b}.$

53. $\left(\dfrac{a+x}{a^2-ax+x^2}-\dfrac{a-x}{a^2+ax+x^2}\right)\div\left(\dfrac{a^2+x^2}{a^3-x^3}-\dfrac{a^2-x^2}{a^3+x^3}\right).$

54. $\dfrac{\dfrac{x-1}{3}+\dfrac{x-1}{x-2}}{\dfrac{x+2}{4}+\dfrac{x+2}{x-3}}\div\dfrac{\dfrac{x+3}{7}-\dfrac{x+3}{x+4}}{\dfrac{x-2}{3}+\dfrac{x-2}{x-1}}.$ 55. $\dfrac{x-2+\dfrac{6}{x+3}}{x-4+\dfrac{12}{x+3}}\times\dfrac{\dfrac{x+3}{4}-\dfrac{x+3}{x+1}}{\dfrac{x-3}{7}+\dfrac{x-3}{x-4}}.$

56. $(x+2)\left\{1+\dfrac{6(x+2)}{x^2-x-6}\right\}\left\{1-\dfrac{5(x+1)}{x^2+3x+2}\right\}.$

57. $(1+a)^2\div\left\{1+\dfrac{a}{1-a+\dfrac{a}{1+a+a^2}}\right\}.$

58. $\dfrac{\{ax^2+(b-c)x-f\}^2-\{ax^2+(b+c)x-f\}^2}{\{ax^2+(b+e)x-f\}^2-\{ax^2+(b-e)x-f\}^2}.$

MISCELLANEOUS EXAMPLES IV.

[*The following Examples for revision are arranged in groups under different headings; each group illustrates one or more of the principal rules and processes already discussed, and for the most part the Examples present more variety and difficulty than those of the same type which have appeared in previous exercises.*]

Substitutions and Brackets.

1. Find the value of $\dfrac{a+\sqrt{a^2+b^2}}{a^3-2b(a^2-b^2)}$ when $a=-4$, $b=-3$.

2. When $a=1$, $b=-1$, $c=2$ evaluate the expression

$$\sqrt{3a^3(b-c)+3b^3(c-a)+3c^3(a-b)}.$$

3. Simplify

$$a(b-c)^3-a(b-c)(2b^2-bc+2c^2)+(ab+ac)(b^2-c^2);$$

and find its value when $a=1$, $b=2$, $c=3$.

4. Find the value of

$$\sqrt[3]{\{5(b^2-c^2)-a^2\}}+\sqrt[4]{3\{a(a^2-c^2)-1\}}$$

when $a=4$, $b=5$, $c=3$.

5. Find the value of

$$\sqrt{(x^2+y^3+z)(x-y-3z)}\div\sqrt[3]{xy^3z^2}$$

when $x=-1$, $y=-3$, $z=1$.

6. When $a=0$, $b=2$, $x=1$, $y=-3$, $z=5$, find the numerical value of

(1) $(x-y)^3-3a(x-y)^2+3b(x^3-y^3)$;

(2) $(x-a)^3-b^2(x-y+z)+\sqrt{(bx^2-axy+y^2+z^2)}$.

7. If $x=6$, $y=7$, $z=8$, find the value of

(1) $x\left\{\dfrac{x}{7}-\dfrac{2}{3}\left(\dfrac{y}{4}+1\right)\right\}-\dfrac{2}{7}\left\{y(x+1)+\dfrac{1}{2}(x^2-2y)-\dfrac{7x}{3}\right\}$;

(2) $x\left\{y\left(z-\dfrac{1}{z}\right)\right\}-\dfrac{y}{z}\left\{y-3\left(\dfrac{x}{3}-xz^2\right)\right\}-2xz\left\{-y\left(1+\dfrac{y}{2xz^2}\right)\right\}$.

8. Evaluate $\dfrac{2a-b\{c-a(b-c)\}}{a^2(b+c)+b^2(c+a)} \cdot \sqrt{\dfrac{a+bc}{b^2+4ac}}$,

when $a=2$, $b=-1$, $c=1$.

9. Find the value of

$$\frac{a-[b-\{a-(b-c)\}-\sqrt{2a^2+b^2+c^2}]}{\sqrt{c^4+a^2b^2+2a^3c}},$$

when $a=-1$, $b=5$, $c=3$.

10. When $a=4$, $b=-2$, $c=\frac{3}{2}$, $d=-1$, find the value of

(1) $a^3-b^3-(a-b)^3-11(3b+2c)\left(2c^2-\dfrac{d^2}{2}\right)$;

(2) $\sqrt[3]{4c^2-a(a-2b-d)}-\sqrt[3]{b^4c+11b^3d^2}$.

11. Find the value of

$$l-m-3\left[\left\{l-m+lm\left(\frac{1}{l}+\frac{1}{m}\right)\right\}^2-4l^2\right]$$

when $l=\frac{2}{3}$, $m=-\frac{1}{3}$.

12. When $a=1$, $b=-\frac{1}{2}$, $c=0$, evaluate

$$\frac{a-[b-c-\{2a-2b-\frac{1}{2}(3c-b)\}]}{a-\dfrac{b}{c-\dfrac{1}{a}}}.$$

13. Simplify

$$42\left\{\frac{4x-3y}{6}-\frac{3}{7}\left(x-\frac{4}{3}y\right)\right\}-56\left\{\frac{1}{7}(3x-2y)-\frac{3}{8}\left(\frac{16}{3}x-y\right)\right\};$$

and find its value when $x=\frac{1}{6}$, $y=-\frac{3}{2}$.

14. If $x=6$, $y=7$, $z=8$ find the value of

(1) $\dfrac{z}{x}\left\{y-\left(2z-\dfrac{x}{z}\right)\right\}-\dfrac{y}{z}\left[\dfrac{z^2}{x}-3\left\{1+\dfrac{2z^3}{3xy}\right\}+\dfrac{x}{2}\right]$;

(2) $3y\left\{\dfrac{2yz}{3}-\left(\dfrac{y}{5}-1\right)\right\}+\dfrac{y^2}{5}\left[3-z\left\{10-\dfrac{x}{2}(y-7)\right\}\right]$.

Resolution into Factors.

(*On Arts.* 128–132.)

Resolve into two or more factors :

15. $x^2+21x+108$.
16. $a^2+6a-91$.
17. $x^2-20xy+96y^2$.
18. $a^2b^2-14ab-51$.
19. c^3+c^2-156c.
20. $m^2n-6mn^2+9n^3$.
21. $p^4-p^2q^2-56q^4$.
22. $d^4-4d^2c^2-45c^4$.
23. $x^3y-x^2y^2-42xy^3$.
24. $m^2+28m+195$.
25. $210-a-a^2$.
26. $57+16pq-p^2q^2$.
27. x^4+27x^2+176.
28. a^4+7a^2-98.
29. $c^2+54c+729$.
30. $72+xy-x^2y^2$.
31. $a^4+9a^2x^2+14x^4$.
32. $p^2-3pq-108q^2$.
33. $2a^6+2a^3-264$.
34. $x^4-2x^3-63x^2$.
35. $b^2c^2+5bc-84$.
36. $z^2+34z+289$.
37. $a^2-22ac+57c^2$.
38. $y^3z+6y^2z-91yz$.
39. $2+x^3-3x^6$.
40. $2a^2b^2+ab-15$.
41. $9p^2-24p+16$.
42. $35+12mn+m^2n^2$.
43. $119-10c-c^2$.
44. $6x^3-5x^4+x^5$.
45. $6m^2+7m-3$.
46. $4a^2-8ab-5b^2$.
47. $6p^2-13pq+2q^2$
48. $20x^2-9xz-20z^2$.
49. $8x^4+2x^2-15$.
50. $12y^2-30y+12$.
51. $12(a^2b^2-1)+7ab$.
52. $2(a^4b^2+5)-9a^2b$.
53. $21x^2+2y(5x-8y)$.
54. $3(6m^2-5n^2)+17mn$.

(*On Arts.* 133–137.)

Resolve into two or more factors :

55. $c^2-a^2-b^2+2ab$.
56. $a^2+2bc-b^2-c^2$.
57. $125x^3+27y^3$.
58. a^3b^3+343.
59. $512b^3-a^6$.
60. $a^2-4(x-y)^2$.
61. $2mn+m^2-1+n^2$.
62. $8c^4-2c^2(d+c)^2$.
63. $(a^2b^2-1)^2-x^2+2xy-y^2$.
64. $1-64m^6$.
65. $p^3+1000p^3q^3$.
66. $6561-a^4$.
67. $x^4-2x^2-y^2-z^2+2yz+1$.
68. $a^2-16(b-c)^2$.
69. $c-d-4(c-d)^3$.
70. p^2-16q^2+p-4q.
71. $2+128(a+b)^3$.
72. $x+3y+x^3+27y^3$.

(Miscellaneous Factors.)

Resolve into two or more factors:

73. $x^3+x^2y+xy^2+y^3$.

74. $acx^2+bcx-adx-bd$.

75. $14-5a-a^2$.

76. $98x^4-7x^2y^2-y^4$.

77. $51-14a-a^2$.

78. $1-(m^2+p^2)-2mp$.

79. $ab(x^2+1)-x(a^2+b^2)$.

80. $9b^2-6bc-16+c^2$.

81. $x^2c^3-c^3+x^2-1$.

82. $3x^2-2ab-x(b-6a)$.

83. $m^3-n^3-(x^2-mn)(m-n)$.

84. $a(b^2+c^2-a^2)+b(a^2+c^2-b^2)$.

85. $x^7-x^3+8x^4-8$.

86. Express in factors the square root of

$$(x^2+8x+7)(2x^2-x-3)(2x^2+11x-21).$$

87. Find the expression whose square is

$$(2x^2-xy-15y^2)(4x^2-25y^2)(2x^2-11xy+15y^2).$$

Highest Common Factor and Lowest Common Multiple.

88. Find the lowest common multiple of

$$13ab^2(x^3-3a^2x+2a^3),\quad 65a^3b(x^2+ax-2a^2),\quad 25b^3(x^2-a^2)^2.$$

89. Find the highest common factor of

$$2(x^4+9)-5x^2(x+1),\quad 2x^3(2x-9)+81(x-1).$$

90. Find the expression of lowest dimensions which is divisible by each of the following expressions:

$$(2x^4+4x^3)(x^2+2x-8),\quad (2x^3-4x^2)(x^2-2x-8),$$
$$(x^2-4x)(x^2+2x-8).$$

91. Find the H.C.F. and the L.C.M. of the three expressions

$$a(a+c)-b(b+c),\quad b(b+a)-c(c+a),\quad c(c+b)-a(a+b).$$

92. Find the divisor of highest dimensions of the expressions

$$(a+b)(a-b)+c(c-2a),\quad (a+c)(a-c)+b(b+2a).$$

93. Find the expression of lowest dimensions such that the L.C.M. of it and $2a^2-3ab+b^2$ is

$$2a^4-3a^3b-a^2b^2+3ab^3-b^4.$$

94. Shew that x^2-4y^2 is the H.C.F. of the expressions

$$x^4-3x^2y^2-4y^4, \quad x^6-64y^6,$$

and $$x^5+32y^5-8x^3y^2+2x^4y+16xy^4-16x^2y^3.$$

95. Find the lowest common multiple of

$$(a-c)^2-(b-c)^2, \quad a^2+b^2-2ac-2bc+2ab, \quad a^4-b^4.$$

96. Shew that the lowest common multiple of

$$a(a-b)^2-ac^2, \quad a^2b-b(b-c)^2, \quad (a+c)^2c-b^2c$$

is $$abc(a^4+b^4+c^4-2b^2c^2-2c^2a^2-2a^2b^2).$$

97. Prove that $$x^4-15x^3+75x^2-145x+84$$

and $$x^4-17x^3+101x^2-247x+210$$

have the same H.C.F and L.C.M. as

$$x^4-13x^3+53x^2-83x+42$$

and $$x^4-19x^3+131x^2-389x+420.$$

Simplification of Fractions.

Simplify

98. $\dfrac{1+x+x^2}{1-x^3}+\dfrac{x-x^2}{(1-x)^3}.$

99. $\dfrac{2x-7}{(x-3)^2}-\dfrac{2(x+2)}{x^2-9}.$

100. $\dfrac{1}{2x^2-\dfrac{1}{2}}+\dfrac{1}{(2x+1)^2}.$

101. $\dfrac{(x+1)^3-(x-1)^3}{3x^3+x}.$

102. $\dfrac{1}{(1-x)^2}+\dfrac{2}{1-x^2}+\dfrac{1}{(1+x)^2}.$

103. $\dfrac{(x^3-2x)^2-(x^2-2)^2}{(x-1)(x+1)(x^2-2)^2}.$

104. $\dfrac{1}{6x-2}-\dfrac{1}{2\left(x-\dfrac{1}{3}\right)}-\dfrac{1}{1-3x}.$

105. $\dfrac{x}{9}+\dfrac{2}{3}+\dfrac{4}{x-6}-\dfrac{2}{3}\cdot\dfrac{1}{1-\dfrac{6}{x}}.$

106. $\dfrac{x}{x^2-y^2}-\dfrac{1}{x-y}+\dfrac{1}{x+y}+\dfrac{1}{x}-\dfrac{1}{y}+\dfrac{x^2-xy+y^2}{xy(x-y)}.$

107. $\left(x-y-\dfrac{4y^2}{x-y}\right)\left(x+y-\dfrac{4x^2}{x+y}\right)\div\left\{3(x+y)-\dfrac{8xy}{x-y}\right\}.$

108. $\dfrac{a^4+b^4+ab(a^2+b^2)}{(a+b)^2}-\dfrac{a^4+b^4-ab(a^2+b^2)}{(a-b)^2}+\dfrac{12a^2b^2}{(a+b)^2-(a-b)^2}.$

109. $\dfrac{(ac+bd)^3-(ad+bc)^3}{(a-b)(c-d)}-\dfrac{(ac+bd)^3+(ad+bc)^3}{(a+b)(c+d)}.$

110. $\left[\left(\dfrac{x}{a}\right)^2+\left(\dfrac{z-x}{b}\right)^2\right]\div\left[\dfrac{z^2}{a^2+b^2}+\dfrac{a^2+b^2}{a^2b^2}\left(x-\dfrac{za^2}{a^2+b^2}\right)^2\right].$

111. $\dfrac{x^4-(x-1)^2}{(x^2+1)^2-x^2}+\dfrac{x^2-(x^2-1)^2}{x^2(x+1)^2-1}+\dfrac{x^2(x-1)^2-1}{x^4-(x+1)^2}.$

112. $\dfrac{\dfrac{1+x}{1-x}+\dfrac{4x}{1+x^2}+\dfrac{8x}{1-x^2}-\dfrac{1-x}{1+x}}{\dfrac{1+x^2}{1-x^2}+\dfrac{4x^2}{1+x^4}-\dfrac{1-x^2}{1+x^2}}.$ 113. $\left\{\dfrac{f}{g-\dfrac{g^2}{f}}+\dfrac{g}{f-\dfrac{f^2}{g}}\right\}\times\dfrac{1}{\dfrac{f^2}{g}-\dfrac{g^2}{f}}.$

114. $\dfrac{1}{x-\dfrac{2}{x+\dfrac{1}{2}}}\times\dfrac{1}{2+\dfrac{1}{x}}\div\dfrac{x}{2x-\dfrac{x+4}{x+1}}.$

115. $\dfrac{x}{1+\dfrac{x}{1-x+\dfrac{x}{1+x}}}\div\dfrac{1+x+x^2}{1+3x+3x^2+2x^3}.$

116. $\dfrac{ab}{a+b}\left(3c+\dfrac{b}{a}\right)-\dfrac{b^2}{(a+b)^3}(a^2+b^2)-2a\left(\dfrac{b}{a+b}\right)^3.$

117. $\dfrac{\dfrac{(a+b)^2+(a-b)^2}{b-a}-(a+b)}{\dfrac{1}{b-a}-\dfrac{1}{a+b}}\div\dfrac{(a+b)^3+(b-a)^3}{(a+b)^2-(a-b)^2}.$

118. $\left\{\dfrac{c-b}{(a-b)(a-c)}-\dfrac{c-a}{(b-c)(b-a)}+\dfrac{b-a}{(c-a)(c-b)}\right\}$

$$\div\frac{2(a^2+b^2+c^2-bc-ca-ab)}{(a-b)(b-c)(c-a)}.$$

CHAPTER XXIII.

HARDER EQUATIONS.

186. In this chapter we propose to give a miscellaneous collection of equations. Some of these will serve as a useful exercise for revision of the methods already explained in previous chapters; but we also add others presenting more difficulty, the solution of which will often be facilitated by some special artifice.

The following examples worked in full will sufficiently illustrate the most useful methods.

Example 1. Solve $\dfrac{6x-3}{2x+7}=\dfrac{3x-2}{x+5}$.

Multiplying up, we have

$$(6x-3)(x+5)=(3x-2)(2x+7),$$
$$6x^2+27x-15=6x^2+17x-14;$$
$$\therefore\ 10x=1;$$
$$\therefore\quad x=\frac{1}{10}.$$

Note. By a simple reduction many equations can be brought to the form in which the above equation is given. When this is the case, the necessary simplification is readily completed by multiplying up, or "multiplying across," as it is sometimes called.

Example 2. Solve $\dfrac{8x+23}{20}-\dfrac{5x+2}{3x+4}=\dfrac{2x+3}{5}-1$.

Multiply by 20, and we have

$$8x+23-\frac{20(5x+2)}{3x+4}=8x+12-20.$$

By transposition, $$31=\frac{20(5x+2)}{3x+4}.$$

Multiplying across, $$93x+124=20(5x+2),$$
$$84=7x;$$
$$\therefore\quad x=12.$$

When two or more fractions have the same denominator hey should be taken together and simplified.

Example 3. Solve $\frac{13-2x}{x+3}+\frac{23x+8\frac{1}{3}}{4x+5}=\frac{16-\frac{1}{4}x}{x+3}+4.$

By transposition, we have

$$\frac{23x+8\frac{1}{3}}{4x+5}-4=\frac{16-\frac{1}{4}x-13+2x}{x+3}.$$

$$\therefore \quad \frac{7x-\frac{35}{3}}{4x+5}=\frac{3+\frac{7x}{4}}{x+3}.$$

Multiplying across, we have

$$7x^2-\frac{35x}{3}+21x-35=12x+7x^2+15+\frac{35x}{4}.$$

$$-\frac{137x}{12}=50;$$

$$\therefore \quad x=-\frac{600}{137}.$$

Example 4. Solve $\frac{x-8}{x-10}+\frac{x-4}{x-6}=\frac{x-5}{x-7}+\frac{x-7}{x-9}.$

This equation might be solved by clearing of fractions, but the ·ork would be very laborious. The solution will be much simplified y proceeding as follows:

Transposing, $\qquad \frac{x-8}{x-10}-\frac{x-5}{x-7}=\frac{x-7}{x-9}-\frac{x-4}{x-6}.$

Simplifying each side *separately*, we have

$$\frac{(x-8)(x-7)-(x-5)(x-10)}{(x-10)(x-7)}=\frac{(x-7)(x-6)-(x-4)(x-9)}{(x-9)(x-6)};$$

$$\therefore \quad \frac{x^2-15x+56-(x^2-15x+50)}{(x-10)(x-7)}=\frac{x^2-13x+42-(x^2-13x+36)}{(x-9)(x-6)};$$

$$\therefore \quad \frac{6}{(x-10)(x-7)}=\frac{6}{(x-9)(x-6)}.$$

Hence, since the numerators are equal, the denominators must be ·qual;

ıat is, $\qquad (x-10)(x-7)=(x-9)(x-6),$

$$x^2-17x+70=x^2-15x+54;$$

$$\therefore \quad 16=2x;$$

$$\therefore \quad x=8$$

The above equation may also be solved very neatly by the following artifice.

The equation may be written in the form

$$\frac{(x-10)+2}{x-10}+\frac{(x-6)+2}{x-6}=\frac{(x-7)+2}{x-7}+\frac{(x-9)+2}{x-9};$$

whence we have

$$1+\frac{2}{x-10}+1+\frac{2}{x-6}=1+\frac{2}{x-7}+1+\frac{2}{x-9};$$

which gives $$\frac{1}{x-10}+\frac{1}{x-6}=\frac{1}{x-7}+\frac{1}{x-9}.$$

Transposing, $$\frac{1}{x-10}-\frac{1}{x-7}=\frac{1}{x-9}-\frac{1}{x-6};$$

$$\therefore \quad \frac{3}{(x-10)(x-7)}=\frac{3}{(x-9)(x-6)},$$

and the solution may be completed as before.

Example 5. Solve $\dfrac{5x-64}{x-13}-\dfrac{2x-11}{x-6}=\dfrac{4x-55}{x-14}-\dfrac{x-6}{x-7}.$

We have $5+\dfrac{1}{x-13}-\left(2+\dfrac{1}{x-6}\right)=4+\dfrac{1}{x-14}-\left(1+\dfrac{1}{x-7}\right);$

$$\therefore \quad \frac{1}{x-13}-\frac{1}{x-6}=\frac{1}{x-14}-\frac{1}{x-7}.$$

The solution may now be completed as before, and we obtain $x=10$.

EXAMPLES XXIII. a.

1. $\dfrac{x+4}{3x-8}=\dfrac{x+5}{3x-7}.$

2. $\dfrac{3x+1}{3(x-2)}=\dfrac{x-2}{x-1}.$

3. $\dfrac{7-5x}{1+x}=\dfrac{11-15x}{1+3x}.$

4. $\dfrac{3(7+6x)}{2+9x}=\dfrac{35+4x}{9+2x}.$

5. $\dfrac{6x+13}{15}-\dfrac{3x+5}{5x-25}=\dfrac{2x}{5}.$

6. $\dfrac{6x+8}{2x+1}-\dfrac{2x+38}{x+12}=1.$

7. $\dfrac{3x-1}{2x-1}-\dfrac{4x-2}{3x-1}=\dfrac{1}{6}.$

8. $\dfrac{x+25}{x-5}=\dfrac{2x+75}{2x-15}.$

9. $\dfrac{x}{x+2}+\dfrac{4}{x+6}=1.$

10. $\dfrac{6x+7}{9x+6}=\dfrac{1}{12}+\dfrac{5x-5}{12x+8}.$

11. $\dfrac{2x-5}{5}+\dfrac{x-3}{2x-15}=\dfrac{4x-3}{10}-1\tfrac{1}{10}.$

12. $\dfrac{4(x+3)}{9}=\dfrac{8x+37}{18}-\dfrac{7x-29}{5x-12}.$

13. $\dfrac{(2x-1)(3x+8)}{6x(x+4)}-1=0.$

14. $\dfrac{2x+5}{5x+3}-\dfrac{2x+1}{5x+2}=0.$

15. $\dfrac{4}{x+3}-\dfrac{2}{x+1}=\dfrac{5}{2x+6}-\dfrac{2\frac{1}{2}}{2x+2}.$

16. $\dfrac{7}{x-4}-\dfrac{60}{5x-30}=\dfrac{10\frac{1}{2}}{3x-12}-\dfrac{8}{x-6}.$

17. $\dfrac{3}{4-2x}+\dfrac{30}{8(1-x)}=\dfrac{3}{2-x}+\dfrac{5}{2-2x}.$

18. $\dfrac{25-\dfrac{x}{3}}{x+1}+\dfrac{16x+4\frac{1}{5}}{3x+2}=5+\dfrac{23}{x+1}.$

19. $\dfrac{30+6x}{x+1}+\dfrac{60+8x}{x+3}=14+\dfrac{48}{x+1}.$

20. $\dfrac{x}{x-2}-\dfrac{x+1}{x-1}=\dfrac{x-8}{x-6}-\dfrac{x-9}{x-7}.$

21. $\dfrac{x+5}{x+4}-\dfrac{x-6}{x-7}=\dfrac{x-4}{x-5}-\dfrac{x-15}{x-16}.$

22. $\dfrac{x-7}{x-9}-\dfrac{x-9}{x-11}=\dfrac{x-13}{x-15}-\dfrac{x-15}{x-17}.$

23. $\dfrac{x+3}{x+6}-\dfrac{x+6}{x+9}=\dfrac{x+2}{x+5}-\dfrac{x+5}{x+8}.$

24. $\dfrac{x+2}{x}+\dfrac{x-7}{x-5}-\dfrac{x+3}{x+1}=\dfrac{x-6}{x-4}.$

25. $\dfrac{4x-17}{x-4}+\dfrac{10x-13}{2x-3}=\dfrac{8x-30}{2x-7}+\dfrac{5x-4}{x-1}.$

26. $\dfrac{5x-8}{x-2}+\dfrac{6x-44}{x-7}-\dfrac{10x-8}{x-1}=\dfrac{x-8}{x-6}.$

27. $\dfrac{2x-3}{.3x-.4}=\dfrac{.4x-.6}{.06x-.07}.$

28. $\dfrac{x-2}{.0\dot{5}}-\dfrac{x-4}{.0625}=56.$

29. $.08\dot{3}(x-.625)=.\dot{0}\dot{9}(x-.59375).$

30. $(2x+1.5)(3x-2.25)=(2x-1.125)(3x+1.25).$

31. $\dfrac{.3x-1}{.5x-.4}=\dfrac{.5+1.2x}{2x-.1}.$

32. $\dfrac{1-1.4x}{.2+x}=\dfrac{.7(x-1)}{.1-.5x}.$

33. $\dfrac{(.3x-2)(.3x-1)}{.2x-1}-\dfrac{1}{6}(.3x-2)=.4x-2.$

Literal Equations.

187. In the equations we have discussed hitherto the coefficients have been numerical quantities, but equations often involve *literal* coefficients. [Art. 6.] These are supposed to be known, and will appear in the solution.

Example 1. Solve $(x+a)(x+b)-c(a+c)=(x-c)(x+c)+ab.$

Multiplying out, we have

$$x^2+ax+bx+ab-ac-c^2=x^2-c^2+ab\,;$$

whence

$$ax+bx=ac,$$

$$(a+b)x=ac\,;$$

$$\therefore\quad x=\frac{ac}{a+b}.$$

Example 2. Solve $$\frac{a}{x-a}-\frac{b}{x-b}=\frac{a-b}{x-c}.$$

Simplifying the left side, we have

$$\frac{a(x-b)-b(x-a)}{(x-a)(x-b)}=\frac{a-b}{x-c}.$$

$$\frac{(a-b)x}{(x-a)(x-b)}=\frac{a-b}{x-c}\,;$$

$$\therefore\quad \frac{x}{(x-a)(x-b)}=\frac{1}{x-c}.$$

Multiplying across, $$x^2-cx=x^2-ax-bx+ab,$$

$$ax+bx-cx=ab,$$

$$(a+b-c)x=ab\,;$$

$$\therefore\quad x=\frac{ab}{a+b-c}.$$

EXAMPLES XXIII. b.

Solve the equations:

1. $ax-2b=5bx-3a.$

2. $a^2(x-a)+b^2(x-b)=abx.$

3. $x^2+a^2=(b-x)^2.$

4. $(x-a)(x+b)=(x-a+b)^2.$

5. $a(x-2)+2x=6+a.$

6. $m^2(m-x)-mnx=n^2(n+x).$

7. $(a+x)(b+x)=x(x-c).$

8. $(a-b)(x-a)=(a-c)(x-b).$

9. $\dfrac{2x+3a}{x+a}=\dfrac{2(3x+2a)}{3x+a}.$

10. $\dfrac{2(x-b)}{3x-c}=\dfrac{2x+b}{3(x-c)}.$

11. $\frac{1}{a}-\frac{1}{x}=\frac{1}{x}-\frac{1}{b}$.

12. $\frac{2}{3}\left(\frac{x}{a}+1\right)=\frac{3}{4}\left(\frac{x}{a}-1\right)$.

13. $\frac{a}{x}=c(a-b)+\frac{b}{x}$.

14. $\frac{9a}{b}-\frac{3x}{b}=\frac{4b}{a}-\frac{2x}{a}$.

15. $\frac{x-a}{b-x}=\frac{x-b}{a-x}$.

16. $\frac{x-a}{2}=\frac{(x-b)^2}{2x-a}$.

17. $\frac{1}{4}x(x-a)-\left(\frac{x+a}{2}\right)^2=\frac{2a}{3}\left(x-\frac{a}{2}\right)$.

18. $(a+b)x^2-a(bx+a^2)=bx(x-a)+ax(x-b)$.

19. $b(a+x)-(a+x)(b-x)=x^2+\frac{bc^2}{a}$.

20. $b(a-x)-\frac{a}{b}(b+x)^2+ab\left(\frac{x}{b}+1\right)^2=0$.

21. $x^2+a(2a-x)-\frac{3b^2}{4}=\left(x-\frac{b}{2}\right)^2+a^2$.

22. $(2x-a)\left(x+\frac{2a}{3}\right)=4x\left(\frac{a}{3}-x\right)-\frac{1}{2}(a-4x)(2a+3x)$.

23. $\frac{x-a+b}{x-a}+\frac{x-b}{x-2b}=\frac{x}{x-b}+\frac{x-a}{x-a-b}$.

24. $\left(\frac{x}{a}-3\right)\left(\frac{3x}{a}-1\right)-\frac{1}{a^2}(x-2a)(2x-a)=\left(\frac{x}{a}-1\right)^2-1$.

25. $\frac{b(x+a)}{x^2-b^2}+\frac{2x+3b-a}{x+b}=\frac{2(x^2+bx-b^2)}{x^2-b^2}$.

Example 3. Solve $ax+by=c$ ....(1),

$a'x+b'y=c'$ ....(2).

The notation here first used is one that the student will frequently meet with in the course of his reading. In the first equation we choose certain letters as the coefficients of x and y, and we choose *corresponding letters with accents* to denote corresponding quantities in the second equation. There is no necessary connection between the values of a and a', and they are as different as a and b; but it is often convenient to use the same letter thus slightly varied to mark some common meaning of such letters, and thereby assist the memory. Thus a, a' have a common property as being coefficients of x; b, b' as being coefficients of y.

Sometimes instead of accents letters are used with a *suffix*, such as a_1, a_2, a_3; b_1, b_2, b_3, etc.

To return to the equations $ax+by=c$ ....(1),

$$a'x+b'y=c' \quad\text{..................................(2).}$$

Multiply (1) by b' and (2) by b. Thus

$$ab'x+bb'y=b'c,$$

$$a'bx+bb'y=bc';$$

by subtraction,
$$(ab'-a'b)x=b'c-bc';$$

$$\therefore\quad x=\frac{b'c-bc'}{ab'-a'b} \quad\text{........................(3).}$$

As previously explained in Art. 104, we might obtain y by substituting this value of x in *either* of the equations (1) or (2); but y is more conveniently found by eliminating x, as follows:

Multiplying (1) by a' and (2) by a, we have

$$aa'x+a'by=a'c,$$

$$aa'x+ab'y=ac';$$

by subtraction,
$$(a'b-ab')y=a'c-ac';$$

$$\therefore\quad y=\frac{a'c-ac'}{a'b-ab'},$$

or, changing signs in the terms of the denominator so as to have the same denominator as in (3),

$$y=\frac{ac'-a'c}{ab'-a'b}, \quad\text{and}\quad x=\frac{b'c-bc'}{ab'-a'b}.$$

Example 4. Solve
$$\frac{x-a}{c-a}+\frac{y-b}{c-b}=1 \quad\text{..................................(1),}$$

$$\frac{x+a}{c}+\frac{y-a}{a-b}=\frac{a}{c} \quad\text{..................................(2).}$$

From (1) by clearing of fractions, we have

$$x(c-b)-a(c-b)+y(c-a)-b(c-a)=(c-a)(c-b),$$

$$x(c-b)+y(c-a)=ac-ab+bc-ab+c^2-ac-bc+ab,$$

$$x(c-b)+y(c-a)=c^2-ab \quad\text{............................(3).}$$

Again, from (2), we have

$$x(a-b)+a(a-b)+cy-ca=a(a-b)$$

$$x(a-b)+cy=ac \quad\text{..............................(4).}$$

Multiply (3) by c and (4) by $c-a$ and subtract,

$$x\{c(c-b)-(c-a)(a-b)\}=c^3-abc-ac(c-a),$$

$$x(c^2-ac+a^2-ab)=c(c^2-ab-ac+a^2);$$

$$\therefore\quad x=c;$$

and therefore from (4)
$$y=b.$$

EXAMPLES XXIII. c.

Solve the equations :

1. $ax+by=l,$
   $bx+ay=m.$

2. $lx+my=n,$
   $px+qy=r.$

3. $ax=by,$
   $bx+ay=c.$

4. $ax+by=a^2,$
   $bx+ay=b^2.$

5. $x+ay=a',$
   $ax+a'y=1.$

6. $px-qy=r,$
   $rx-py=q.$

7. $\frac{x}{a}+\frac{y}{b}=\frac{1}{ab},$
   $\frac{x}{a'}-\frac{y}{b'}=\frac{1}{a'b'}.$

8. $\frac{x}{a}-\frac{y}{b}=0,$
   $bx+ay=4ab.$

9. $\frac{3x}{a}+\frac{2y}{b}=3,$
   $\frac{9x}{a}-\frac{6y}{b}=3.$

10. $qx-rb=p(a-y),$
    $\frac{qx}{a}+r=p\left(1+\frac{y}{b}\right).$

11. $\frac{x}{m}+\frac{y}{m'}=1,$
    $\frac{x}{m'}-\frac{y}{m}=1.$

12. $px+qy=0,$
    $lx+my=n.$

13. $(a-b)x=(a+b)y,$
    $x+y=c.$

14. $(a-b)x+(a+b)y=2a^2-2b^2.$
    $(a+b)x-(a-b)y=4ab.$

15. $\frac{x}{a}+\frac{y}{b}=1,$
    $\frac{x}{3a}+\frac{y}{6b}=\frac{2}{3}.$

16. $\frac{x}{a}+\frac{y}{b}=2,$
    $\frac{x}{a'}=\frac{y}{b'}.$

17. $\frac{x}{a}-\frac{y}{b}=1,$
    $\frac{x}{b}+\frac{y}{a}=\frac{a}{b}.$

18. $\frac{m}{l}x+\frac{l}{m}y=\left(\frac{1}{l}+\frac{1}{m}\right)(m^2+l^2),$
    $(x+y)(m^2+l^2)=2(m^3+l^3)+ml(x+y).$

19. $bx+cy=a+b,$
    $ax\left(\frac{1}{a-b}-\frac{1}{a+b}\right)+cy\left(\frac{1}{b-a}-\frac{1}{b+a}\right)=\frac{2a}{a+b}.$

20. $(a-b)x+(a+b)y=2(a^2-b^2), \quad ax-by=a^2+b^2.$

21. $x\left(a-b+\frac{ab}{a-b}\right)=y\left(a+b-\frac{ab}{a+b}\right). \quad x+y=2a^3.$

CHAPTER XXIV.

HARDER PROBLEMS.

188. In previous chapters we have given collections of problems which lead to simple equations. We add here a few examples of somewhat greater difficulty.

Example 1. A grocer buys 15 lbs. of figs and 28 lbs. of currants for \$2.60; by selling the figs at a loss of 10 per cent., and the currants at a gain of 30 per cent., he clears 30 cents on his outlay; how much per pound did he pay for each?

Let x, y denote the number of cents in the price of a pound of figs and currants respectively; then the outlay is

$$15x + 28y \text{ cents.}$$

Therefore $$15x + 28y = 260 \quad\text{.................................}(1).$$

The loss upon the figs is $\frac{1}{10} \times 15x$ cents, and the gain upon the currants is $\frac{3}{10} \times 28y$ cents; therefore the total gain is

$$\frac{42y}{5} - \frac{3x}{2} \text{ cents;}$$

$$\therefore \quad \frac{42y}{5} - \frac{3x}{2} = 30 \quad\text{.................................}(2).$$

From (1) and (2) we find that $x = 8$, and $y = 5$; that is the figs cost 8*c.* a pound, and the currants cost 5*c.* a pound.

Example 2. At what time between 4 and 5 o'clock will the minute-hand of a watch be 13 minutes in advance of the hour-hand?

Let x denote the required number of minutes after 4 o'clock; then, as the minute-hand travels twelve times as fast as the hour-hand, the hour-hand will move over $\frac{x}{12}$ minute divisions in x minutes. At 4 o'clock the minute-hand is 20 divisions behind the hour-hand, and finally the minute-hand is 13 divisions in advance; therefore the minute-hand moves over $20 + 13$, or 33 divisions more than the hour-hand.

Hence $$x = \frac{x}{12} + 33,$$

$$\frac{11}{12}x = 33;$$

$$\therefore \quad x = 36.$$

Thus the time is 36 minutes past 4.

If the question be asked as follows: "At what *times* between 4 and 5 o'clock will there be 13 minutes between the two hands?" we must also take into consideration the case when the minute-hand is 13 divisions *behind* the hour-hand. In this case the minute-hand gains 20 – 13, or 7 divisions.

Hence $$x = \frac{x}{12} + 7,$$

which gives $$x = 7\frac{7}{11}.$$

Therefore the *times* are $7\frac{7'}{11}$ past 4, and 36′ past 4.

Example 3. Two persons A and B start simultaneously from two places, c miles apart, and walk in the same direction. A travels at the rate of p miles an hour, and B at the rate of q miles; how far will A have walked before he overtakes B?

Suppose A has walked x miles, then B has walked $x - c$ miles.

A walking at the rate of p miles an hour will travel x miles in $\frac{x}{p}$ hours; and B will travel $x - c$ miles in $\frac{x-c}{q}$ hours: these two times being equal, we have

$$\frac{x}{p} = \frac{x-c}{q},$$

$$qx = px - pc;$$

whence $$x = \frac{pc}{p-q}.$$

Therefore A has travelled $\frac{pc}{p-q}$ miles.

Example 4. A train travelled a certain distance at a uniform rate. Had the speed been 6 miles an hour more, the journey would have occupied 4 hours less; and had the speed been 6 miles an hour less, the journey would have occupied 6 hours more. Find the distance.

Let the speed of the train be x miles per hour, and let the time occupied be y hours; then the distance traversed will be represented by xy miles.

On the first supposition the speed per hour is $x+6$ miles, and the time taken is $y-4$ hours. In this case the distance traversed will be represented by $(x+6)(y-4)$ miles.

On the second supposition the distance traversed will be represented by $(x-6)(y+6)$ miles.

All these expressions for the distance must be equal;

$$\therefore\ xy=(x+6)(y-4)=(x-6)(y+6).$$

From these equations we have

$$xy=xy+6y-4x-24,$$

or $$6y-4x=24 \text{ ....(1)};$$

and $$xy=xy-6y+6x-36,$$

or $$6x-6y=36 \text{ ....(2)}.$$

From (1) and (2) we obtain $x=30$, $y=24$.

Hence the distance is 720 miles.

Example 5. A person invests \$3770, partly in 3 per cent. Stock at \$102, and partly in Railway Stock at \$84 which pays a dividend of $4\frac{1}{2}$ per cent.: if his income from these investments is \$136.25 per annum, what sum does he invest in each?

Let x denote the number of dollars invested in 3 per cent., y the number of dollars invested in Railway Stock; then

$$x+y=3770 \text{ ....(1)}.$$

The income from 3 per cent. Stock is $\$\frac{3x}{102}$, or $\$\frac{x}{34}$; and that from Railway Stock is $\$\frac{4\frac{1}{2}y}{84}$, or $\$\frac{3y}{56}$.

Therefore $$\frac{x}{34}+\frac{3y}{56}=136\tfrac{1}{4} \text{ ....(2)}.$$

From (2) $$x+\frac{51}{28}y=4632\tfrac{1}{2};$$

therefore by subtracting (1)

$$\frac{23}{28}y=862\tfrac{1}{2};$$

whence $$y=28\times 37\tfrac{1}{2}=1050;$$

and from (1) $$x=2720.$$

Therefore he invests \$2720 in 3 per cent. Stock, and \$1050 in Railway Stock.

EXAMPLES XXIV.

1. A sum of \$100 is divided among a number of persons; if the number had been increased by one-fourth each would have received a half-dollar less: find the number of persons.

2. I bought a certain number of marbles at four for a cent; I kept one-fifth of them, and sold the rest at three for a cent, and gained a cent: how many did I buy?

3. I bought a certain number of articles at five for six cents; if they had been eleven for twelve cents, I should have spent six cents less: how many did I buy?

4. A boy lost a half of his marbles and one more; in the next game he lost a half of what remained and two more, and he then had only three left: how many had he at first?

5. A purse containing gold coins is divided amongst three persons. The first receives a half of them and one more, the second a half of the remainder and one more, and the third had six. How many coins did the purse contain?

6. A number of two digits exceeds five times the sum of its digits by 9, and its ten-digit exceeds its unit-digit by 1: find the number.

7. The sum of the digits of a number less than 100 is 6; if the digits be reversed the resulting number will be less by 18 than the original number: find it.

8. A man being asked his age replied, "If you take 2 years from my present age the result will be double my wife's age, and 3 years ago her age was one-third of what mine will be in 12 years." What were their ages?

9. At what time between one and two o'clock are the hands of a watch first at right angles?

10. At what time between 3 and 4 o'clock is the minute-hand one minute ahead of the hour hand?

11. When are the hands of a clock together between the hours of 6 and 7?

12. It is between 2 and 3 o'clock, and in 10 minutes the minute-hand will be as much before the hour-hand as it is now behind it: what is the time?

13. At an election the majority was 162, which was three-elevenths of the whole numbers of voters: what was the number of the votes on each side?

14. A certain number of persons paid a bill; if there had been 10 more each would have paid \$2 less; if there had been 5 less each would have paid \$2.50 more: find the number of persons, and what each had to pay.

15. A man spends \$100 in buying two kinds of silk at \$4.50 and \$4 a yard; by selling it at \$4.25 per yard he gains 2 per cent: how much of each did he buy?

16. Ten years ago the sum of the ages of two sons was one-third of their father's age: one is two years older than the other, and the present sum of their ages is fourteen years less than their father's age: how old are they?

17. *A* and *B* start from the same place walking at different rates; when *A* has walked 15 miles *B* doubles his pace, and 6 hours later passes *A*: if *A* walks at the rate of 5 miles an hour, what is *B*'s rate at first?

18. A basket of oranges is emptied by one person taking half of them and one more, a second person taking half of the remainder and one more, and a third person taking half of the remainder and six more. How many did the basket contain at first?

19. A person swimming in a stream which runs $1\frac{1}{2}$ miles per hour, finds that it takes him four times as long to swim a mile up the stream as it does to swim the same distance down: at what rate does he swim?

20. At what *times* between 7 and 8 o'clock will the hands of a watch be at right angles to each other? When will they be in the same straight line?

21. The denominator of a fraction exceeds the numerator by 4; and if 5 is taken from each, the sum of the reciprocal of the new fraction and four times the original fraction is 5: find the original fraction.

22. Two persons start at noon from towns 60 miles apart. One walks at the rate of four miles an hour, but stops $2\frac{1}{2}$ hours on the way; the other walks at the rate of 3 miles an hour without stopping: when and where will they meet?

23. *A*, *B*, and *C* travel from the same place at the rates of 4, 5, and 6 miles an hour respectively; and *B* starts 2 hours after *A*. How long after *B* must *C* start in order that they may overtake *A* at the same instant?

24. A dealer bought a horse, expecting to sell it again at a price that would have given him 10 per cent. profit on his purchase; but he had to sell it for \$50 less than he expected, and he then found that he had lost 15 per cent. on what it cost him: what did he pay for the horse?

25. A man walking from a town, *A*, to another, *B*, at the rate of 4 miles an hour, starts one hour before a coach travelling 12 miles an hour, and is picked up by the coach. On arriving at *B*, he finds that his coach journey has lasted 2 hours: find the distance between *A* and *B*.

26. What is the property of a person whose income is \$1140, when one-twelfth of it is invested at 2 per cent., one-half at 3 per cent., one-third at $4\frac{1}{2}$ per cent., and the remainder pays him no dividend?

27. A person spends one-third of his income, saves one-fourth, and pays away 5 per cent. on the whole as interest at $7\frac{1}{2}$ per cent. on debts previously incurred, and then has \$110 remaining: what was the amount of his debts?

28. Two vessels contain mixtures of wine and water; in one there is three times as much wine as water, in the other, five times as much water as wine. Find how much must be drawn off from each to fill a third vessel which holds seven gallons, in order that its contents may be half wine and half water.

29. There are two mixtures of wine and water, one of which contains twice as much water as wine, and the other three times as much wine as water. How much must there be taken from each to fill a pint cup, in which the water and the wine shall be equally mixed?

30. Two men set out at the same time to walk, one from A to B, and the other from B to A, a distance of a miles. The former walks at the rate of p miles, and the latter at the rate of q miles an hour: at what distance from A will they meet?

31. A train on the North Western line passes from London to Birmingham in 3 hours; a train on the Great Western line which is 15 miles longer, travelling at a speed which is less by 1 mile per hour, passes from one place to the other in $3\frac{1}{2}$ hours: find the length of each line.

32. Coffee is bought at 36 cents and chicory at 9 cents per lb.: in what proportion must they be mixed that 10 per cent. may be gained by selling the mixture at 33 cents per lb.?

33. A man has one kind of coffee at a cents per pound, and another at b cents per pound. How much of each must he take to form a mixture of $a-b$ lbs., which he can sell at c cents a pound without loss?

34. A man spends c dollars in buying two kinds of silk at a cents and b cents a yard respectively; he could have bought 3 times as much of the first and half as much of the second for the same money: how many yards of each did he buy?

35. A man rides one-third of the distance from A to B at the rate of a miles an hour, and the remainder at the rate of $2b$ miles an hour. If he had travelled at a uniform rate of $3c$ miles an hour, he could have ridden from A to B and back again in the same time. Prove that

$$\frac{2}{c}=\frac{1}{a}+\frac{1}{b}.$$

36. A, B, C are three towns forming a triangle. A man has to walk from one to the next, ride thence to the next, and drive thence to his starting point. He can walk, ride, and drive a mile in a, b, c minutes respectively. If he starts from B he takes $a+c-b$ hours, if he starts from C he takes $b+a-c$ hours, and if he starts from A he takes $c+b-a$ hours. Find the length of the circuit.

CHAPTER XXV.

QUADRATIC EQUATIONS.

189. SUPPOSE the following problem were proposed for solution:

A dealer bought a number of sheep for \$280. If he had bought four less each would have cost \$8 more: how many did he buy?

We should proceed thus:

Let x = the number of sheep; then $\frac{280}{x}$ = the number of dollars each cost.

If he had bought 4 less he would have had $x-4$ sheep, and each would have cost $\frac{280}{x-4}$ dollars.

$$\therefore \quad 8+\frac{280}{x}=\frac{280}{x-4};$$

whence
$$x(x-4)+35(x-4)=35x;$$

$$\therefore \quad x^2-4x+35x-140=35x;$$

$$\therefore \quad x^2-4x=140.$$

Here we have an equation which involves the *square* of the unknown quantity; and in order to complete the solution of the problem we must discover a method of solving such equations.

190. DEFINITION. An equation which contains the square of the unknown quantity, but no higher power, is called a **quadratic equation,** or an **equation of the second degree.**

If the equation contains both the square and the first power of the unknown it is called an *adfected* quadratic; if it contains only the square of the unknown it is said to be a *pure* quadratic.

Thus $2x^2-5x=3$ is an adfected quadratic,

and $5x^2=20$ is a pure quadratic.

191. A pure quadratic may be considered as a simple equation in which the *square* of the unknown quantity is to be found.

Example. Solve $\frac{9}{x^2-27}=\frac{25}{x^2-11}$.

Multiplying up, $9x^2-99=25x^2-675$;

$\therefore\ 16x^2=576$;

$\therefore\ x^2=36$;

and taking the square root of these equals, we have

$$x=\pm 6.$$

Note. We prefix the double sign to the number on the right-hand side for the reason given in Art. 117.

192. In extracting the square root of the two sides of the equation $x^2=36$, it might seem that we ought to prefix the double sign to the quantities on both sides, and write $\pm x=\pm 6$. But an examination of the various cases shews this to be unnecessary. For $\pm x=\pm 6$ gives the four cases:

$$+x=+6,\quad +x=-6,\quad -x=+6,\quad -x=-6,$$

and these are all included in the two already given, namely, $x=+6$, $x=-6$. Hence, when we extract the square root of the two sides of an equation, it is sufficient to put the double sign before the square root of *one* side.

193. The equation $x^2=36$ is an instance of the simplest form of quadratic equations. The equation $(x-3)^2=25$ may be solved in a similar way; for taking the square root of both sides, we have two *simple* equations,

$$x-3=\pm 5.$$

Taking the upper sign, $x-3=+5$, whence $x=8$;

taking the lower sign, $x-3=-5$, whence $x=-2$.

$\therefore$ the solution is $x=8$, or -2.

Now the given equation, $(x-3)^2=25$

may be written $x^2-6x+(3)^2=25$,

or $x^2-6x=16$.

Hence, by retracing our steps, we learn that the equation

$$x^2-6x=16.$$

can be solved by first adding $(3)^2$ or 9 to each side, and then extracting the square root; and the reason why we add 9 to each side is that this quantity added to the left side makes it a *perfect square.*

Now whatever the quantity a may be,

$$x^2+2ax+a^2=(x+a)^2,$$

and

$$x^2-2ax+a^2=(x-a)^2;$$

so that if a trinomial is a perfect square, and *its highest power*, x^2, *has unity for its coefficient*, we must always have the term without x equal to the square of half the coefficient of x. If, therefore, the terms in x^2 and x are given, the square may be completed by adding the square of half the coefficient of x.

Note. When an expression is a perfect square, the *square terms* are always *positive*. [Art. 114, Note.] Hence, if necessary, the coefficient of x^2 must be made equal to $+1$ before completing the square.

Example 1. Solve $x^2+14x=32$.

The square of half 14 is $(7)^2$.

$$\therefore \quad x^2+14x+(7)^2=32+49;$$

that is,

$$(x+7)^2=81;$$

$$\therefore \quad x+7=\pm 9;$$

$$\therefore \quad x=-7+9, \text{ or } -7-9;$$

$$\therefore \quad x=2, \text{ or } -16.$$

Example 2. Solve $7x=x^2-8$.

Transpose so as to have the terms involving x on one side, and the square term positive.

Thus

$$x^2-7x=8.$$

Completing the square, $x^2-7x+\left(\frac{7}{2}\right)^2=8+\frac{49}{4}$;

that is,

$$\left(x-\frac{7}{2}\right)^2=\frac{81}{4};$$

$$\therefore \quad x-\frac{7}{2}=\pm\frac{9}{2};$$

$$\therefore \quad x=\frac{7}{2}\pm\frac{9}{2};$$

$$\therefore \quad x=8, \text{ or } -1.$$

Note. We do not work out $\left(\frac{7}{2}\right)^2$ on the left-hand side.

EXAMPLES XXV. a.

1. $5(x^2+5)=6x^2$.
2. $3x^2=4(x^2-4)$.
3. $x^2+22x=75$.
4. $x^2+24x=25$.
5. $x^2=10x-21$.
6. $(9+x)(9-x)=17$.
7. $x^2+3x=18$.
8. $x^2+5x=14$.
9. $x^2-5x-36=0$.

10. $x^2=x+72.$ 11. $x^2-341=20x.$ 12. $9x-x^2+220=0.$

13. $68-x^2=13x.$ 14. $x+156=x^2.$ 15. $187=x^2+6x.$

16. $23x=120+x^2.$ 17. $42+x^2=13x.$ 18. $22x+23-x^2=0.$

19. $x^2-\frac{2}{3}x=32.$ 20. $x^2+\frac{4}{15}x=\frac{1}{5}.$ 21. $x^2-\frac{7}{6}x-\frac{1}{2}=0.$

22. $\frac{19}{5}x=\frac{4}{5}-x^2.$ 23. $\frac{3}{5}(x+6)(x-2)=\frac{2}{3}\left(62\frac{1}{10}+\frac{18x}{5}\right).$

194. We have shewn that the square may readily be completed when the coefficient of x^2 is unity. All cases may be reduced to this by dividing the equation throughout by the coefficient of x^2.

Example 1. Solve $$32-3x^2=10x.$$

Transposing, $$3x^2+10x=32.$$

Divide throughout by 3, so as to make the coefficient of x^2 unity.

Thus $$x^2+\frac{10}{3}x=\frac{32}{3};$$

completing the square, $$x^2+\frac{10}{3}x+\left(\frac{5}{3}\right)^2=\frac{32}{3}+\frac{25}{9};$$

that is, $$\left(x+\frac{5}{3}\right)^2=\frac{121}{9};$$

$$\therefore\quad x+\frac{5}{3}=\pm\frac{11}{3};$$

$$\therefore\quad x=-\frac{5}{3}\pm\frac{11}{3}=2,\text{ or }-5\tfrac{1}{3}.$$

Note. We do not add $\left(\frac{10}{6}\right)^2$ but $\left(\frac{5}{3}\right)^2$ to the left-hand side.

Example 2. Solve $$5x^2+11x=12.$$

Dividing by 5, $$x^2+\frac{11}{5}x=\frac{12}{5};$$

completing the square, $$x^2+\frac{11}{5}x+\left(\frac{11}{10}\right)^2=\frac{12}{5}+\frac{121}{100};$$

that is, $$\left(x+\frac{11}{10}\right)^2=\frac{361}{100};$$

$$\therefore\quad x+\frac{11}{10}=\pm\frac{19}{10};$$

$$\therefore\quad x=-\frac{11}{10}\pm\frac{19}{10}=\frac{4}{5},\text{ or }-3.$$

195. We see then that the following are the steps required for solving an adfected quadratic equation:

(1) If necessary, simplify the equation so that the terms in x^2 and x are on one side of the equation, and the term without x on the other.

(2) Make the coefficient of x^2 unity and positive by dividing throughout by the coefficient of x^2.

(3) Add to each side of the equation the square of half the coefficient of x.

(4) Take the square root of each side.

(5) Solve the resulting simple equations.

196. In the examples which follow some preliminary reduction and simplification may be necessary.

Example 1. Solve $\dfrac{3x-2}{2x-3}=\dfrac{5x}{x+4}-2.$

Simplifying, $\dfrac{3x-2}{2x-3}=\dfrac{3x-8}{x+4};$

multiplying across, $3x^2+10x-8=6x^2-25x+24;$

that is, $-3x^2+35x=32.$

Dividing by -3, $x^2-\dfrac{35}{3}x=-\dfrac{32}{3};$

completing the square, $x^2-\dfrac{35}{3}x+\left(\dfrac{35}{6}\right)^2=\dfrac{1225}{36}-\dfrac{32}{3};$

that is, $\left(x-\dfrac{35}{6}\right)^2=\dfrac{841}{36};$

$$\therefore\quad x-\frac{35}{6}=\pm\frac{29}{6};$$

$$\therefore\quad x=10\tfrac{2}{3},\text{ or }1.$$

Example 2. Solve $7(x+2a)^2+3a^2=5a(7x+23a).$

Simplifying, $7x^2+28ax+28a^2+3a^2=35ax+115a^2,$

that is, $7x^2-7ax=84a^2.$

Whence $x^2-ax=12a^2;$

completing the square, $x^2-ax+\left(\dfrac{a}{2}\right)^2=12a^2+\dfrac{a^2}{4};$

that is, $\left(x-\dfrac{a}{2}\right)^2=\dfrac{49a^2}{4};$

$$\therefore\quad x-\frac{a}{2}=\pm\frac{7a}{2};$$

$$\therefore\quad x=4a,\text{ or }-3a.$$

197. Sometimes there is *only one solution.* Thus if

$$x^2-2x+1=0, \text{ then } (x-1)^2=0,$$

whence $x=1$ is the only solution. Nevertheless, in this and similar cases we find it convenient to say that the quadratic has *two equal roots.*

EXAMPLES XXV. b.

1. $5x^2+14x=55$.
2. $3x^2+121=44x$.
3. $25x=6x^2+21$.
4. $8x^2+x=30$.
5. $3x^2+35=22x$.
6. $x+22-6x^2=0$.
7. $15=17x+4x^2$.
8. $21+x=2x^2$.
9. $9x^2-143-6x=0$.
10. $12x^2=29x-14$.
11. $20x^2=12-x$.
12. $19x=15-8x^2$.
13. $21x^2+22x+5=0$.
14. $50x^2-15x=27$.
15. $18x^2-27x-26=0$.
16. $5x^2=8x+21$.
17. $15x^2-2ax=a^2$.
18. $21x^2=2ax+3a^2$.
19. $6x^2=11kx+7k^2$.
20. $12x^2+23kx+10k^2=0$.
21. $12x^2-cx-20c^2=0$.
22. $2(x-3)=3(x+2)(x-3)$.
23. $(x+1)(2x+3)=4x^2-22$.
24. $(3x-5)(2x-5)=x^2+2x-3$.
25. $\dfrac{5x+7}{x-1}=3x+2$.
26. $\dfrac{5x-1}{x+1}=\dfrac{3x}{2}$.
27. $\dfrac{3x-8}{x-2}=\dfrac{5x-2}{x+5}$.
28. $\dfrac{3x-1}{4x+7}=1-\dfrac{6}{x+7}$.
29. $\dfrac{5x-7}{7x-5}=\dfrac{x-5}{2x-13}$.
30. $\dfrac{x+3}{2x-7}-\dfrac{2x-1}{x-3}=0$.
31. $\dfrac{1}{1+x}-\dfrac{1}{3-x}=\dfrac{6}{35}$.
32. $\dfrac{x+4}{x-4}+\dfrac{x-2}{x-3}=6\frac{1}{3}$.
33. $\dfrac{1}{3-x}-\dfrac{4}{5}=\dfrac{1}{9-2x}$.
34. $\dfrac{4}{x-1}-\dfrac{5}{x+2}=\dfrac{3}{x}$.
35. $\dfrac{5}{x-2}-\dfrac{4}{x}=\dfrac{3}{x+6}$.
36. $\dfrac{x-2}{x-3}+\dfrac{3x-11}{x-4}=\dfrac{4x+13}{x+1}$.
37. $\dfrac{1}{2x-5a}+\dfrac{5}{2x-a}=\dfrac{2}{a}$.
38. $\dfrac{2}{3x-2c}+\dfrac{3}{2x-3c}=\dfrac{7}{2c}$.
39. $\dfrac{a^2b}{x^2}+\left(1+\dfrac{b}{x}\right)a=2b+\dfrac{a^2}{x}$.

198. From the preceding examples it appears that after suitable reduction and transposition every quadratic equation can be written in the form

$$ax^2+bx+c=0,$$

where a, b, c may have any numerical values whatever. If therefore we can solve this quadratic we can solve any.

Transposing, $$ax^2+bx=-c;$$

dividing by a, $$x^2+\frac{b}{a}x=-\frac{c}{a}.$$

Completing the square by adding to each side $\left(\frac{b}{2a}\right)^2$,

$$x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{c}{a};$$

that is, $$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2};$$

extracting the square root,

$$x+\frac{b}{2a}=\frac{\pm\sqrt{(b^2-4ac)}}{2a};$$

$$x=\frac{-b\pm\sqrt{(b^2-4ac)}}{2a}.$$

199. Instead of going through the process of completing the square in each particular example, we may now make use of this general formula, adapting it to the case in question by substituting the values of a, b, c.

Example. Solve $5x^2+11x-12=0$.

Here $a=5$, $b=11$, $c=-12$.

$$\therefore\ x=\frac{-11\pm\sqrt{(11)^2-4\,.\,5(-12)}}{10}$$

$$=\frac{-11\pm\sqrt{361}}{10}=\frac{-11\pm 19}{10}=\frac{4}{5},\ \text{or}\ -3,$$

which agrees with the solution of Example 2, Art. 194.

200. In the result $$x=\frac{-b+\sqrt{(b^2-4ac)}}{2a},$$

it must be remembered that the expression $\sqrt{(b^2-4ac)}$ is the square root of the compound quantity b^2-4ac, *taken as a whole.* We cannot simplify the solution unless we know the numerical values of a, b, c. It may sometimes happen that these values do not make b^2-4ac a perfect square. In such a case the exact numerical solution of the equation cannot be determined.

Example 1. Solve $5x^2 - 15x + 11 = 0$.

We have
$$x = \frac{15 \pm \sqrt{(-15)^2 - 4 \cdot 5 \cdot 11}}{2 \cdot 5}$$
$$= \frac{15 \pm \sqrt{5}}{10} \quad \text{..................................(1)}$$

Now $\sqrt{5} = 2{\cdot}236$ approximately.
$$\therefore \quad x = \frac{15 \pm 2{\cdot}236}{10} = 1{\cdot}7236, \text{ or } 1{\cdot}2764.$$

These solutions are correct only to four places of decimals, and neither of them will be found to *exactly* satisfy the equation.

Unless the *numerical* values of the unknown quantity are required it is usual to leave the roots in the form (1).

Example 2. Solve $x^2 - 3x + 5 = 0$.

We have
$$x = \frac{3 \pm \sqrt{(-3)^2 - 4 \cdot 1 \cdot 5}}{2}$$
$$= \frac{3 \pm \sqrt{9 - 20}}{2}$$
$$= \frac{3 \pm \sqrt{-11}}{2}.$$

Now there is no quantity, positive or negative, whose square is negative (Art. 110). Therefore it is impossible to find any quantity exactly or approximately to represent the square root of -11. Thus there is no real value of x which satisfies the equation. In such a case the roots are said to be *imaginary* or *impossible*. A reference to the general formula of Art. 198 will shew that the roots of a quadratic $ax^2 + bx + c = 0$ are always imaginary when $b^2 - 4ac$ is negative.

Note. If the equation $x^2 - 3x + 5 = 0$ is treated graphically, as explained in Art. 330, it will be found that the graph never meets the axis of x. In other words there is no numerical value of x which makes the expression $x^2 - 3x + 5$ equal to zero.

[Chap. XXXV., Arts. 325–330 *may be read here.*]

201. Solution by Factors. The following method of solution will sometimes be found shorter than either of the methods given.

Consider the equation $x^2 + \frac{7}{3}x = 2$.

Clearing of fractions, $3x^2 + 7x - 6 = 0$ ....(1);

by resolving the left-hand side into factors we have
$$(3x - 2)(x + 3) = 0.$$

Now if *either* of the factors $3x-2$, $x+3$ is zero their product is zero. Hence the quadratic equation is satisfied by either of the suppositions

$$3x-2=0, \text{ or } x+3=0.$$

Thus the roots are $\frac{2}{3}$, -3.

It appears from this that *when a quadratic equation has been simplified and brought to the form of equation* (1), its solution can always be readily obtained if the expression on the left-hand side can be resolved into factors. Each of these factors equated to zero gives a simple equation, and a corresponding root of the quadratic.

Example 1. Solve $2x^2-ax+2bx=ab$.

Transposing, *so as to have all the terms on one side of the equation*, we have

$$2x^2-ax+2bx-ab=0.$$

Now $\quad 2x^2-ax+2bx-ab=x(2x-a)+b(2x-a)$

$$=(2x-a)(x+b).$$

Therefore $\quad (2x-a)(x+b)=0;$

whence $\quad 2x-a=0, \text{ or } x+b=0.$

$$\therefore \quad x=\frac{a}{2}, \text{ or } -b.$$

Example 2. Solve $2(x^2-6)=3(x-4)$.

We have $\quad 2x^2-12=3x-12;$

that is, $\quad 2x^2=3x \quad \text{.................................} (1).$

Transposing, $\quad 2x^2-3x=0,$

$$x(2x-3)=0.$$

$$\therefore \quad x=0, \text{ or } 2x-3=0.$$

Thus the roots are 0, $\frac{3}{2}$.

Note. In equation (1) above we might have divided both sides by x and obtained the simple equation $2x=3$, whence $x=\frac{3}{2}$, which is *one* of the solutions of the given equation. But the student must be particularly careful to notice that whenever an x, or a factor containing x, is removed by division from every term of an equation it must not be neglected, since the equation is satisfied by $x=0$, which is therefore one of the roots.

202. There are some equations which are not really quadratics, but which may be solved by the methods explained in this chapter.

Example 1. Solve $x^4-13x^2+36=0.$

By resolution into factors, $(x^2-9)(x^2-4)=0$;

$$\therefore \quad x^2-9=0, \text{ or } x^2-4=0;$$

that is, $x^2=9$, or 4,

and $x=\pm 3$, or ± 2.

Example 2. Solve $x^2+3x-\dfrac{20}{x^2+3x}=8.$

Write y for x^2+3x, then we have

$$y-\frac{20}{y}=8,$$

or $$y^2-8y-20=0.$$

From this quadratic $y=10$, or -2;

$$\therefore \quad x^2+3x=10, \text{ or } -2.$$

Thus we have *two* quadratics to solve, and finally we obtain $x=-5,\ 2$; or $-1,\ -2$.

EXAMPLES XXV. c.

Solve by Art. 200, and verify graphically by Art. 330:

1. $3x^2=15-4x.$ **2.** $2x^2+7x=15.$ **3.** $2x^2+7-9x=0.$

4. $x^2=3x+5.$ **5.** $5x^2+4+21x=0.$ **6.** $x^2+11=7x.$

7. $8x^2=x+7.$ **8.** $5x^2=17x-10.$ **9.** $35+9x-2x^2=0.$

10. $3x^2=x+1.$ **11.** $3x^2+5x=2.$ **12.** $2x^2+5x-33=0.$

Solve by resolution into factors:

13. $6x^2=7+x.$ **14.** $21+8x^2=26x.$

15. $26x-21+11x^2=0.$ **16.** $5x^2+26x+24=0.$

17. $4x^2=\dfrac{4}{15}x+3.$ **18.** $x^2-2=\dfrac{23}{12}x.$

19. $7x^2=28-96x.$ **20.** $96x^2=4x+15.$

21. $25x^2=5x+6.$ **22.** $35-4x=4x^2.$

23. $12x^2-11ax=36a^2.$ **24.** $12x^2+36a^2=43ax.$

25. $35b^2=9x^2+6bx.$ **26.** $36x^2-35b^2=12bx.$

27. $x^2-2ax+4ab=2bx.$ **28.** $x^2-2ax+8x=16a.$

29. $3x^2-2ax-bx=0.$ **30.** $ax^2+2x=bx.$

Solve as explained in Art. 202:

31. $4 = 5x^2 - x^4$.

32. $x^4 + 36 = 13x^2$.

33. $x^6 + 7x^3 = 8$.

34. $x^6 - 19x^3 = 216$.

35. $16\left(x^2 + \dfrac{1}{x^2}\right) = 257$.

36. $x^2 + \dfrac{a^2b^2}{x^2} = a^2 + b^2$.

37. $x^3(19 + x^3) = 216$.

38. $(x^2+2)^2 + 198 = 29(x^2+2)$.

39. $x^2 - x + \dfrac{72}{x^2 - x} = 18$.

40. $x(x - 2a) = \dfrac{8a^4}{x^2 - 2ax} + 7a^2$.

202_A. The method of solution by factors is applicable to equations of higher degree than the second.

For example, if

$$(x-2)(x+1)(x+2)=0,$$

the equation must be satisfied by each of the values which satisfy the equations

$$x-2=0, \quad x+1=0, \quad x+2=0.$$

Thus the roots are $x = 2, -1, -2$.

Example. Solve the equation $3x^3 + 5x^2 = 3x + 5$.

Putting the equation in the form

$$3x^3 + 5x^2 - 3x - 5 = 0,$$

we have $$x^2(3x+5) - (3x+5) = 0,*$$

or $$(x^2 - 1)(3x+5) = 0;$$

that is, $$(x+1)(x-1)(3x+5) = 0;$$

whence $$x+1=0, \text{ or } x-1=0, \text{ or } 3x+5=0.$$

Thus the roots are $-1, 1, -\frac{5}{3}$.

Note. At the stage marked with an asterisk we might have divided throughout by $3x+5$, but in so doing the factor must be equated to zero to furnish one root of the equation.

202_B. If one root of an equation is known, or can be obtained by trial, a corresponding factor of the first degree can be removed. When this is done we have left an equation of lower degree than the original equation.

Example. Solve the equation

$$x^3 - 3x^2 - 6x + 16 = 0.$$

By trial it will be found that the left-hand side vanishes when $x = 2$.

Hence $x=2$ is one root of the equation and corresponding to this root we have a factor $x-2$; the equation may now be written

$$x^2(x-2)-x(x-2)-8(x-2)=0;$$

or
$$(x^2-x-8)(x-2)=0.$$

Removing the factor $x-2$, we have

$$x^2-x-8=0;$$

whence
$$x=\frac{1\pm\sqrt{33}}{2}.$$

Thus the three roots are $2,\ \dfrac{1+\sqrt{33}}{2},\ \dfrac{1-\sqrt{33}}{2}.$

EXAMPLES XXV. d.

Solve the following equations by the method of factors.

1. $x^3+x^2-x-1=0.$ **2.** $x^3-2x^2-x+2=0.$

3. $x^3-4x=x^2-4.$ **4.** $x^3+7x^2+7x-15=0.$

5. $x^3-3x-2=0.$ **6.** $x^4+2x=3x^2.$ **7.** $x^3+30=19x.$

Solve the following equations having given one root in each case.

8. $x^3-39x+70=0.$ $[x=5.]$ **9.** $x^3-37x-84=0.$ $[x=-3.]$

10. $x^3-12a^2x=16a^3.$ $[x=4a.]$ **11.** $x^4+432a^3x=108a^2x^2.$ $[x=6a.]$

Solve the following equations by the method of Art. 200, giving the roots to two places of decimals.

12. $x^2+2x=3{\cdot}2.$ **13.** $x^2-3x-3{\cdot}51=0.$

14. $x^2+x=1{\cdot}0956.$ **15.** $x^2-36x+323{\cdot}7=0.$

16. $x^2-7x+6{\cdot}035=0.$ **17.** $x^2-5{\cdot}5x+7{\cdot}3776=0.$

18. Find two values of x which will make $x(3x-1)$ equal to ${\cdot}362$, giving each value to the nearest hundredth.

19. Find to the nearest tenth the values of x which will make $2x(2-x)$ equal to $1{\cdot}73$.

20. Solve the equation $x^2+ax-a^2=0$. If $a=12$, give the numerical values of the roots to three decimal places.

21. Solve the equation $x(a-x)=c^2$. Give the numerical values of the roots to three decimal places, when $a=16$, $c=6$.

CHAPTER XXVI.

SIMULTANEOUS QUADRATIC EQUATIONS.

203. WE shall now consider some of the most useful methods of solving simultaneous equations, one or more of which may be of a degree higher than the first; but no fixed rules can be laid down which are applicable to all cases.

Example 1. Solve $x+y=15$ ....(1).
$xy=36$ ....(2).

From (1) by squaring, $x^2+2xy+y^2=225$;
from (2) $4xy=144$;
by subtraction, $x^2-2xy+y^2=81$;
by taking the square root, $x-y=\pm 9$.

Combining this with (1) we have to consider the two cases,

$$\left.\begin{aligned}x+y&=15,\\x-y&=9.\end{aligned}\right\}\qquad\left.\begin{aligned}x+y&=15,\\x-y&=-9.\end{aligned}\right\}$$

from which we find $\left.\begin{aligned}x&=12,\\y&=3.\end{aligned}\right\}\qquad\left.\begin{aligned}x&=3,\\y&=12.\end{aligned}\right\}$

Example 2. Solve $x-y=12$ ....(1),
$xy=85$ ....(2).

From (1) $x^2-2xy+y^2=144$;
from (2) $4xy=340$;
by addition, $x^2+2xy+y^2=484$;
by taking the square root, $x+y=\pm 22$.

Combining this with (1) we have the two cases,

$$\left.\begin{aligned}x+y&=22,\\x-y&=12.\end{aligned}\right\}\qquad\left.\begin{aligned}x+y&=-22,\\x-y&=12.\end{aligned}\right\}$$

Whence $\left.\begin{aligned}x&=17,\\y&=5.\end{aligned}\right\}\qquad\left.\begin{aligned}x&=-5,\\y&=-17.\end{aligned}\right\}$ [See Art. 333.]

204. These are the simplest cases that arise, but they are specially important since the solution in a large number of other cases is dependent upon them.

As a rule our object is to solve the proposed equations *symmetrically*, by finding the values of $x+y$ and $x-y$. From the foregoing examples it will be seen that we can always do this as soon as we have obtained the product of the unknowns, and either their sum or their difference

Example 1. Solve $x^2+y^2=74$ ....(1),

$xy=35$ ....(2).

Multiply (2) by 2; then by addition and subtraction we have

$$x^2+2xy+y^2=144,$$
$$x^2-2xy+y^2=\quad 4;$$

Whence
$$x+y=\pm 12,$$
$$x-y=\pm\ 2.$$

We have now four cases to consider; namely,

$$\left.\begin{aligned}x+y&=12,\\ x-y&=2.\end{aligned}\right\}\quad \left.\begin{aligned}x+y&=12,\\ x-y&=-2.\end{aligned}\right\}\quad \left.\begin{aligned}x+y&=-12,\\ x-y&=2.\end{aligned}\right\}\quad \left.\begin{aligned}x+y&=-12,\\ x-y&=-2.\end{aligned}\right\}$$

From which the values of x are $7,\ 5,\ -5,\ -7$; [Compare and the corresponding values of y are $5,\ 7,\ -7,\ -5$. Art. 333.]

Example 2. Solve $x^2+y^2=185$..............................(1),

$x+y=\ 17$..............................(2).

By subtracting (1) from the square of (2) we have

$$2xy=104;$$
$$\therefore\ xy=\ 52..............................(3).$$

Equations (1) and (3) can now be solved by the method of Example 1; and the solution is

$$\left.\begin{aligned}x&=13, \text{ or } 4,\\ y&=4, \text{ or } 13.\end{aligned}\right\}$$

EXAMPLES XXVI. a.

Solve the following equations:

1. $x+y=28,\quad xy=187.$
2. $x+y=51,\quad xy=518.$
3. $x+y=74,\quad xy=1113.$
4. $x-y=5,\quad xy=126.$
5. $x-y=8,\quad xy=513.$
6. $xy=1075,\quad x-y=18.$
7. $xy=923,\quad x+y=84.$
8. $x-y=-8,\quad xy=1353.$
9. $x-y=-22,\quad xy=3848.$

Solve the following equations :

10. $xy = -2193,\ x+y=-8.$

11. $x-y=-18,\ xy=1363.$

12. $xy=-1914,\ x+y=-65.$

13. $x^2+y^2=89,\ xy=40.$

14. $x^2+y^2=170,\ xy=13.$

15. $x^2+y^2=65,\ xy=28.$

16. $x^2+y^2=178,\ x+y=16.$

17. $x+y=15,\ x^2+y^2=125.$

18. $x-y=4,\ x^2+y^2=106.$

19. $x^2+y^2=180,\ x-y=6.$

20. $x^2+y^2=185,\ x-y=3.$

21. $x+y=13,\ x^2+y^2=97.$

22. $x+y=9,\ x^2+xy+y^2=61.$

23. $x-y=3,\ x^2-3xy+y^2=-19.$

24. $x^2-xy+y^2=76,\ x+y=14.$

25. $\frac{1}{10}(x-y)=1,\ x^2-4xy+y^2=52.$

26. $\frac{1}{x}+\frac{1}{y}=2,\ x+y=2.$

27. $\frac{1}{x}+\frac{1}{y}=\frac{7}{12},\ xy=12,$

28. $ax+by=2,\ abxy=1.$

29. $x^2+pxy+y^2=p+2,\ qx^2+xy+qy^2=2q+1.$

205. Any pair of equations of the form

$$x^2 \pm pxy + y^2 = a^2 \quad\text{.........(1),}$$
$$x \pm y = b \quad\text{.........(2),}$$

where p is any numerical quantity, can be reduced to one of the cases already considered; for by squaring (2) and combining with (1), an equation to find xy is obtained; the solution can then be completed by the aid of equation (2).

Example 1. Solve

$$x^3 - y^3 = 999 \quad\text{.........(1),}$$
$$x - y = 3 \quad\text{.........(2),}$$

By division, $x^2+xy+y^2=333$ ....(3);

from (2) $x^2-2xy+y^2=9$;

by subtraction, $3xy=324$,

$$xy=108 \quad\text{.........(4).}$$

From (2) and (4) $\left.\begin{array}{l} x=12, \text{ or } -9, \\ y=9, \text{ or } -12. \end{array}\right\}$

Example 2. Solve

$$x^4+x^2y^2+y^4=2613 \quad\text{.........(1),}$$
$$x^2+xy+y^2=67 \quad\text{.........(2).}$$

Dividing (1) by (2) $x^2-xy+y^2=39$ ....(3).

From (2) and (3) by addition, $x^2+y^2=53$;

by subtraction, $xy=14$;

whence $\left.\begin{array}{l} x=\pm 7,\ \pm 2, \\ y=\pm 2,\ \pm 7. \end{array}\right\}$ [Art. 204, Ex. 1.]

Example 3. Solve
$$\frac{1}{x}-\frac{1}{y}=\frac{1}{3}\qquad\text{.....(1)},$$
$$\frac{1}{x^2}+\frac{1}{y^2}=\frac{5}{9}\qquad\text{.....(2)}.$$

From (1) by squaring, $\frac{1}{x^2}-\frac{2}{xy}+\frac{1}{y^2}=\frac{1}{9}$;

by subtraction, $\frac{2}{xy}=\frac{4}{9}$;

adding to (2), $\frac{1}{x^2}+\frac{2}{xy}+\frac{1}{y^2}=1$;

$$\therefore\quad \frac{1}{x}+\frac{1}{y}=\pm 1.$$

Combining with (1), $\frac{1}{x}=\frac{2}{3}$, or $-\frac{1}{3}$,

$\frac{1}{y}=\frac{1}{3}$, or $-\frac{2}{3}$;

$$\therefore\quad x=\frac{3}{2},\text{ or } -3,\text{ and } y=3,\text{ or } -\frac{3}{2}.$$

EXAMPLES XXVI. b.

Solve the equations :

1. $x^3+y^3=407$, $x+y=11$.
2. $x^3+y^3=637$, $x+y=13$.
3. $x+y=23$, $x^3+y^3=3473$.
4. $x^3-y^3=218$, $x-y=2$.
5. $x-y=4$, $x^3-y^3=988$.
6. $x^3-y^3=2197$, $x-y=13$.
7. $x^4+x^2y^2+y^4=2128$, $x^2+xy+y^2=76$.
8. $x^4+x^2y^2+y^4=2923$, $x^2-xy+y^2=37$.
9. $x^4+x^2y^2+y^4=9211$, $x^2-xy+y^2=61$.
10. $x^4+x^2y^2+y^4=7371$, $x^2-xy+y^2=63$.
11. $\frac{1}{x^2}+\frac{1}{y^2}=\frac{481}{576}$, $\frac{1}{x}+\frac{1}{y}=\frac{29}{24}$,
12. $\frac{1}{x^2}+\frac{1}{y^2}=\frac{61}{900}$, $xy=30$.
13. $\frac{x}{y}+\frac{y}{x}=2\frac{1}{2}$, $x+y=6$.
14. $\frac{x}{y}+\frac{y}{x}=2\frac{16}{21}$, $x-y=4$.
15. $\frac{34}{x^2+y^2}=\frac{15}{xy}$, $x+y=8$.
16. $x^3-y^3=56$, $x^2+xy+y^2=28$.
17. $4(x^2+y^2)=17xy$, $x-y=6$.
18. $x^3+y^3=126$, $x^2-xy+y^2=21$.

Solve the equations :

19. $\dfrac{1}{x^3}+\dfrac{1}{y^3}=1\tfrac{1}{125}$,

$\dfrac{1}{x}+\dfrac{1}{y}=1\tfrac{1}{5}$.

20. $\dfrac{1}{x^3}-\dfrac{1}{y^3}=91$,

$\dfrac{1}{x}-\dfrac{1}{y}=1$.

206. The following method of solution may always be used when the equations are *of the same degree and homogeneous.* [See Art. 24.]

Example. Solve
$$x^2+xy+2y^2=74 \quad\text{.........(1)},$$
$$2x^2+2xy+y^2=73 \quad\text{.........(2)}.$$

Put $y=mx$, and substitute in both equations. Thus
$$x^2(1+m+2m^2)=74 \quad\text{.........(3)}.$$
and
$$x^2(2+2m+m^2)=73 \quad\text{.........(4)}.$$

By division,
$$\frac{1+m+2m^2}{2+2m+m^2}=\frac{74}{73};$$
$$\therefore\quad 73+73m+146m^2=148+148m+74m^2;$$
$$\therefore\quad 72m^2-75m-75=0,$$
or
$$24m^2-25m-25=0;$$
$$\therefore\quad (8m+5)(3m-5)=0;$$
$$\therefore\quad m=-\frac{5}{8},\text{ or }\frac{5}{3}.$$

(i) Take $m=-\dfrac{5}{8}$, and substitute in either (3) or (4).

From (3)
$$x^2\left(1-\frac{5}{8}+\frac{50}{64}\right)=74;$$
$$\therefore\quad x^2=\frac{64\times 74}{74}=64;$$
$$\therefore\quad x=\pm 8;$$
$$\therefore\quad y=mx=-\frac{5}{8}x=\mp 5.$$

(ii) Take $m=\dfrac{5}{3}$; then from (3)
$$x^2\left(1+\frac{5}{3}+\frac{50}{9}\right)=74.$$
$$x^2=\frac{74\times 9}{74}=9;$$
$$\therefore\quad x=\pm 3;$$
$$\therefore\quad y=mx=\frac{5}{3}x=\pm 5.$$

207. When one of the equations is of the first degree and the other of a higher degree, we may from the simple equation find the value of one of the unknowns in terms of the other, and substitute in the second equation.

Example. Solve $3x-4y=5$ ....(1),
$3x^2-xy-3y^2=21$ ....(2).

From (1) we have $x=\dfrac{5+4y}{3}$;

and substituting in (2), $\dfrac{3(5+4y)^2}{9}-\dfrac{y(5+4y)}{3}-3y^2=21$;

$$\therefore\ 75+120y+48y^2-15y-12y^2-27y^2=189;$$
$$9y^2+105y-114=0,$$
$$3y^2+35y-38=0;$$
$$\therefore\ (y-1)(3y+38)=0;$$
$$\therefore\ y=1, \text{ or } -\frac{38}{3};$$

and by substituting in (1), $x=3$, or $-\dfrac{137}{9}$;

208. The examples we have given will be sufficient as a general explanation of the methods to be employed; but in some cases special artifices are necessary.

Example 1. Solve $x^2+4xy+3x=40-6y-4y^2$ ....(1),
$2xy-x^2=3$ ....(2).

From (1) we have $x^2+4xy+4y^2+3x+6y=40$;

that is, $(x+2y)^2+3(\overline{x+2y})-40=0$,

or $(\overline{x+2y}+8)(\overline{x+2y}-5)=0$;

whence $x+2y=-8$, or 5.

(i) Combining $x+2y=5$ with (2) we obtain
$$2x^2-5x+3=0;$$

whence $x=1$, or $\dfrac{3}{2}$;

and by substituting in $x+2y=5$, $y=2$, or $\dfrac{7}{4}$.

(ii) Combining $x+2y=-8$ with (2) we obtain
$$2x^2+8x+3=0;$$

whence $x=\dfrac{-4\pm\sqrt{10}}{2}$; and $y=\dfrac{-12\mp\sqrt{10}}{4}$.

Example 2. Solve $x^2y^2 - 6x = 34 - 3y$ ....(1),

$3xy + y = 2(9 + x)$ ....(2).

From (1) $x^2y^2 - 6x + 3y = 34$;

from (2) $9xy - 6x + 3y = 54$;

by subtraction, $x^2y^2 - 9xy + 20 = 0$,

$$(xy - 5)(xy - 4) = 0;$$

$$\therefore \quad xy = 5, \text{ or } 4.$$

(i) Substituting $xy = 5$ in (2) gives $y - 2x = 3$.

From these equations we obtain $\left.\begin{array}{l} x = 1, \text{ or } -\frac{5}{2}, \\ y = 5, \text{ or } -2. \end{array}\right\}$

(ii) Substituting $xy = 4$ in (2) gives $y - 2x = 6$.

From these equations we obtain $x = \dfrac{-3 \pm \sqrt{17}}{2}$,

and $y = 3 \pm \sqrt{17}$.

EXAMPLES XXVI. c.

Solve the equations:

1. $5x - y = 17$, $xy = 12$.
2. $x^2 + xy = 15$, $y^2 + xy = 10$.
3. $x - y = 10$, $x^2 - 2xy - 3y^2 = 84$.
4. $3x + 2y = 16$, $xy = 10$.
5. $3x - y = 11$, $3x^2 - y^2 = 47$.
6. $x - 3y = 1$, $x^2 - 2xy + 9y^2 = 17$.
7. $x + 2y = 9$, $3y^2 - 5x^2 = 43$.
8. $x^2 + y^2 = 5$, $2xy - y^2 = 3$.
9. $5x + y = 3$, $2x^2 - 3xy - y^2 = 1$.
10. $3x^2 - 5y^2 = 28$, $3xy - 4y^2 = 8$.
11. $3x^2 - y^2 = 23$, $2x^2 - xy = 12$.
12. $x^2 + xy + y^2 = 3\frac{1}{4}$. $2x^2 - 3xy + 2y^2 = 2\frac{3}{4}$.
13. $x^2 - 3xy + y^2 + 1 = 0$, $3x^2 - xy + 3y^2 = 13$.
14. $7xy - 8x^2 = 10$, $8y^2 - 9xy = 18$.
15. $x^2 - 2xy = 21$, $xy + y^2 = 18$.
16. $x^2 + 3xy = 54$, $xy + 4y^2 = 115$.
17. $x^3 + y^3 = 152$, $x^2y + xy^2 = 120$.
18. $x^3 - y^3 = 127$, $x^2y - xy^2 = 42$.
19. $x^3 - y^3 = 208$, $xy(x - y) = 48$.
20. $x^2y^2 + 5xy = 84$, $x + y = 8$.
21. $x^2 + 4y^2 + 80 = 15x + 30y$, $xy = 6$.
22. $9x^2 + y^2 - 63x - 21y + 128 = 0$, $xy = 4$.

CHAPTER XXVII.

Problems leading to Quadratic Equations.

209. We shall now discuss some problems which give rise to quadratic equations.

Example 1. A train travels 300 miles at a uniform rate; if the rate had been 5 miles an hour more, the journey would have taken two hours less: find the rate of the train.

Suppose the train travels at the rate of x miles per hour, then the time occupied is $\frac{300}{x}$ hours.

On the other supposition the time is $\frac{300}{x+5}$ hours;

$$\therefore \quad \frac{300}{x+5}=\frac{300}{x}-2 \quad \text{..........}(1);$$

whence $$x^2+5x-750=0,$$

or $$(x+30)(x-25)=0,$$

$$\therefore \quad x=25, \text{ or } -30.$$

Hence the train travels 25 miles per hour, the negative value being inadmissible.

It will frequently happen that the algebraical statement of the question leads to a result which does not apply to the actual problem we are discussing. But such results can sometimes be explained by a suitable modification of the conditions of the question. In the present case we may explain the negative solution as follows.

Since the values $x=25$ and -30 satisfy the equation (1), if we write $-x$ for x the resulting equation,

$$\frac{300}{-x+5}=\frac{300}{-x}-2 \quad \text{..........}(2),$$

will be satisfied by the values $x=-25$ and 30. Now, by changing signs throughout, equation (2) becomes $\frac{300}{x-5}=\frac{300}{x}+2$;

and this is the algebraical statement of the following question:

A train travels 300 miles at a uniform rate; if the rate had been 5 miles an hour *less*, the journey would have taken two hours *more*: find the rate of the train. The rate is 30 miles an hour.

Example 2. A person selling a horse for \$72 finds that his loss per cent. is one-eighth of the number of dollars that he paid for the horse: what was the cost price?

Suppose that the cost price of the horse is x dollars; then the loss on \$100 is $\$\frac{x}{8}$.

Hence the loss on $\$x$ is $x \times \frac{x}{800}$, or $\frac{x^2}{800}$ dollars;

$\therefore$ the selling price is $x - \frac{x^2}{800}$ dollars.

Hence $$x - \frac{x^2}{800} = 72,$$

or $$x^2 - 800x + 57600 = 0;$$

that is, $$(x-80)(x-720) = 0;$$

$$\therefore \; x = 80, \text{ or } 720;$$

and each of these values will be found to satisfy the conditions of the problem. Thus the cost is either \$80, or \$720.

Example 3. A cistern can be filled by two pipes in $33\frac{1}{3}$ minutes; if the larger pipe takes 15 minutes less than the smaller to fill the cistern, find in what time it will be filled by each pipe singly.

Suppose that the two pipes running singly would fill the cistern in x and $x-15$ minutes. When running together they will fill $\left(\frac{1}{x} + \frac{1}{x-15}\right)$ of the cistern in one minute. But they fill $\frac{1}{33\frac{1}{3}}$, or $\frac{3}{100}$ of the cistern in one minute.

Hence $$\frac{1}{x} + \frac{1}{x-15} = \frac{3}{100},$$

$$100(2x-15) = 3x(x-15),$$

$$3x^2 - 245x + 1500 = 0,$$

$$(x-75)(3x-20) = 0;$$

$$\therefore \; x = 75, \text{ or } 6\tfrac{2}{3}.$$

Thus the smaller pipe takes 75 minutes, the larger 60 minutes. The other solution $6\frac{2}{3}$ is inadmissible.

Example 4. By rowing half the distance and walking the other half, a man can travel 24 miles on a river in 5 hours with the stream, and in 7 hours against the stream. If there were no current, the journey would take $5\frac{2}{3}$ hours: find the rate of his walking, and rowing, and the rate of the stream.

Suppose that the man walks x miles per hour, rows y miles per hour, and that the stream flows at the rate of z miles per hour.

With the current the man rows $y+z$ miles, and against the current $y-z$ miles per hour.

Hence we have the following equations:

$$\frac{12}{x}+\frac{12}{y+z}=5 \qquad\qquad (1),$$

$$\frac{12}{x}+\frac{12}{y-z}=7 \qquad\qquad (2),$$

$$\frac{12}{x}+\frac{12}{y}=5\tfrac{2}{3} \qquad\qquad (3).$$

From (1) and (3) by subtraction, $\dfrac{1}{y}-\dfrac{1}{y+z}=\dfrac{1}{18}$ ....(4).

Similarly, from (2) and (3) $\dfrac{1}{y-z}-\dfrac{1}{y}=\dfrac{1}{9}$ ....(5).

From (4) $18z=y(y+z)$ ....(6);

and from (5) $9z=y(y-z)$ ....(7).

From (6) and (7) by division, $2=\dfrac{y+z}{y-z}$;

whence $y=3z$;

$\therefore$ from (4) $z=1\frac{1}{2}$; and hence $y=4\frac{1}{2}$, $x=4$.

Thus the rates of walking and rowing are 4 miles and $4\frac{1}{2}$ miles per hour respectively; and the stream flows at the rate of $1\frac{1}{2}$ miles per hour.

EXAMPLES XXVII.

1. Find a number whose square diminished by 119 is equal to ten times the excess of the number over 8.

2. A man is five times as old as his son, and the sum of the squares of their ages is equal to 2106: find their ages.

3. The sum of the reciprocals of two consecutive numbers is $\dfrac{15}{56}$: find them.

4. Find a number which when increased by 17 is equal to 60 times the reciprocal of the number.

5. Find two numbers whose sum is 9 times their difference, and the difference of whose squares is 81.

6. The sum of a number and its square is nine times the next highest number: find it.

7. If a train travelled 5 miles an hour faster it would take one hour less to travel 210 miles: what time does it take?

8. Find two numbers the sum of whose squares is 74, and whose sum is 12.

9. The perimeter of a rectangular field is 500 yards, and its area is 14400 square yards: find the length of the sides.

10. The perimeter of one square exceeds that of another by 100 feet; and the area of the larger square exceeds three times the area of the smaller by 325 square feet: find the length of their sides.

11. A cistern can be filled by two pipes running together in $22\frac{1}{2}$ minutes; the larger pipe would fill the cistern in 24 minutes less than the smaller one: find the time taken by each.

12. A man travels 108 miles, and finds that he could have made the journey in $4\frac{1}{2}$ hours less had he travelled 2 miles an hour faster: at what rate did he travel?

13. I buy a number of foot-balls for $100; had they cost a dollar apiece less, I should have had five more for the money: find the cost of each.

14. A boy was sent for 40 cents' worth of eggs. He broke 4 on his way home, and the cost therefore was at the rate of 3 cents more than the market price for 6. How many did he buy?

15. What are eggs a dozen when two more in 24 cents' worth lowers the price 2 cents per dozen?

16. A lawn 50 feet long and 34 feet broad has a path of uniform width round it; if the area of the path is 540 square feet, find its width.

17. A hall can be paved with 200 square tiles of a certain size; if each tile were one inch longer each way it would take 128 tiles: find the length of each tile.

18. In the centre of a square garden is a square lawn; outside this is a gravel walk 4 feet wide, and then a flower border 6 feet wide. If the flower border and lawn together contain 721 square feet, find the area of the lawn.

19. By lowering the price of apples and selling them one cent a dozen cheaper, an applewoman finds that she can sell 60 more than she used to do for 60 cents. At what price per dozen did she sell them at first?

20. Two rectangles contain the same area, 480 square yards. The difference of their lengths is 10 yards, and of their breadths 4 yards: find their sides.

21. There is a number between 10 and 100; when multiplied by the digit on the left the product is 280; if the sum of the digits be multiplied by the same digit the product is 55: required the number.

22. A farmer having sold, at \$75 each, horses which cost him x dollars apiece, finds that he has realized x per cent. profit on his outlay: find x.

23. A tradesman bought a number of yards of cloth for £5; he kept 5 yards and sold the rest at 2*s.* per yard more than he gave, and got £1 more than he originally spent: how many yards did he buy?

24. If a carriage wheel $14\frac{2}{3}$ ft. in circumference takes one second more to revolve, the rate of the carriage per hour will be $2\frac{2}{3}$ miles less: how fast is the carriage travelling?

25. A broker bought as many railway shares as cost him \$1875: he reserved 15, and sold the remainder for \$1740, gaining \$4 a share on their cost price. How many shares did he buy?

26. A and B are two stations 300 miles apart. Two trains start simultaneously from A and B, each to the opposite station. The train from A reaches B nine hours, the train from B reaches A four hours after they meet: find the rate at which each train travels.

27. A train A starts to go from P to Q, two stations 240 miles apart, and travels uniformly. An hour later another train B starts from P, and after travelling for 2 hours, comes to a point that A had passed 45 minutes previously. The pace of B is now increased by 5 miles an hour, and it overtakes A just on entering Q. Find the rates at which they started.

28. A cask P is filled with 50 gallons of water, and a cask Q with 40 gallons of brandy; x gallons are drawn from each cask, mixed and replaced; and the same operation is repeated. Find x when there are $8\frac{7}{8}$ gallons of brandy in P after the second replacement.

29. Two farmers A and B have 30 cows between them; they sell at different prices, but each receives the same sum. If A had sold his at B's price, he would have received \$320; and if B had sold his at A's price, he would have received \$245. How many had each?

30. A man arrives at the railway station nearest to his house $1\frac{1}{2}$ hours before the time at which he had ordered his carriage to meet him. He sets out at once to walk at the rate of 4 miles an hour, and, meeting his carriage when it had travelled 8 miles, reaches home exactly 1 hour earlier than he had originally expected. How far is his house from the station, and at what rate was his carriage driven?

31. P is a point in a line AB of length a. Find AP when $AB \,.\, BP = AP^2$. Explain both solutions.

32. If a straight line 6 cm. in length is divided internally so that the rectangle contained by the whole and one part is equal to the square on the other part, find the segments of the line to the nearest millimetre.

33. A line AB is produced to P so that $AB \,.\, AP = BP^2$ If $AB = 8$ cm., find the lengths of AP and BP to the nearest millimetre.

34. If a line AB of any length is divided externally as in the last Example, shew that

(i) $AB^2 + AP^2 = 3BP^2$; (ii) $(AB + AP)^2 = 5BP^2$.

35. A line AB is produced to P so that $BP^2 = 2AB^2$. If $AB = 3{\cdot}5$ cm., find AP to the nearest millimetre.

36. Find a point P in a straight line AB so that

$$AP(AP - BP) = BP^2.$$

If $AB = 4{\cdot}2$ cm., find AP and BP to the nearest millimetre. By substituting these values verify the truth of the given relation.

37. Divide a straight line 13 centimetres long into two parts so that the rectangle contained by them may be equal to 36 square centimetres.

38. Justify the following graphical solution of the previous Example:

On AB, a line 13 cm. in length, describe a semicircle. At A draw AP perpendicular to AB and 6 cm. in length; through P draw a line PQR to cut the semicircle in Q and R; draw QX, RY perpendicular to AB. Then AB is divided as required either at X or Y. Verify the algebraical solution of Example 37 by actual measurement.

39. Solve the following equations graphically, taking a centimetre as unit and giving the roots to the nearest millimetre.

(i) $x(7 - x) = 12$; (ii) $x^2 - 11x + 30 = 0$;

(iii) $x^2 - 6x + 4 = 0$; (iv) $x^2 + 13 = 8x$.

CHAPTER XXVIII.

HARDER FACTORS.

210. IN Chapter XVII. we have explained several rules for resolving algebraical expressions into factors; in the present chapter we shall continue the subject by discussing cases of greater difficulty.

211. By a slight modification some expressions admit of being written in the form of the difference of two squares, and may then be resolved into factors by the method of Art. 133.

Example 1. Resolve into factors $x^4+x^2y^2+y^4$.

$$\begin{aligned} x^4+x^2y^2+y^4 &= (x^4+2x^2y^2+y^4)-x^2y^2 \\ &= (x^2+y^2)^2-(xy)^2 \\ &= (x^2+y^2+xy)(x^2+y^2-xy) \\ &= (x^2+xy+y^2)(x^2-xy+y^2). \end{aligned}$$

Example 2. Resolve into factors $x^4-15x^2y^2+9y^4$.

$$\begin{aligned} x^4-15x^2y^2+9y^4 &= (x^4-6x^2y^2+9y^4)-9x^2y^2 \\ &= (x^2-3y^2)^2-(3xy)^2 \\ &= (x^2-3y^2+3xy)(x^2-3y^2-3xy). \end{aligned}$$

212. Expressions which can be put into the form $x^3 \pm \frac{1}{y^3}$ may be separated into factors by the rules for resolving the sum or the difference of two cubes. [Art. 136.]

Example 1.
$$\begin{aligned} \frac{8}{a^3}-27b^6 &= \left(\frac{2}{a}\right)^3-(3b^2)^3 \\ &= \left(\frac{2}{a}-3b^2\right)\left(\frac{4}{a^2}+\frac{6b^2}{a}+9b^4\right). \end{aligned}$$

Example 2. Resolve $a^2x^3-\frac{8a^2}{y^3}-x^3+\frac{8}{y^3}$ into four factors.

$$\begin{aligned} a^2x^3-\frac{8a^2}{y^3}-x^3+\frac{8}{y^3} &= x^3(a^2-1)-\frac{8}{y^3}(a^2-1) \\ &= (a^2-1)\left(x^3-\frac{8}{y^3}\right) \\ &= (a+1)(a-1)\left(x-\frac{2}{y}\right)\left(x^2+\frac{2x}{y}+\frac{4}{y^2}\right). \end{aligned}$$

Example 3. Resolve $a^9-64a^3-a^6+64$ into six factors.

$$\begin{aligned}\text{The expression} &= a^3(a^6-64)-(a^6-64)\\ &=(a^6-64)(a^3-1)\\ &=(a^3+8)(a^3-8)(a^3-1)\\ &=(a+2)(a^2-2a+4)(a\quad 2)(a^2+2a+4)(a-1)(a^2+a+1).\end{aligned}$$

Example 4.

$$a(a-1)x^2-(a-b-1)xy-b(b+1)y^2=\{ax-(b+1)y\}\{(a-1)x+by\}.$$

Note. In examples of this kind the coefficients of x and y in the binomial factors can usually be guessed at once, and it only remains to verify the coefficient of the middle term.

213. From Example 2, Art. 52, we see that the quotient of

$$a^3+b^3+c^3-3abc \text{ by } a+b+c \text{ is } a^2+b^2+c^2-bc-ca-ab.$$

Thus $\quad a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-bc-ca-ab)\ldots(1).$

This result is important and should be carefully remembered. We may note that the expression on the left consists of the sum of the cubes of three quantities a, b, c, diminished by 3 times the product abc. Whenever an expression admits of a similar arrangement, the above formula will enable us to resolve it into factors.

Example 1. Resolve into factors $a^3-b^3+c^3+3abc$.

$$\begin{aligned}a^3-b^3+c^3+3abc &= a^3+(-b)^3+c^3-3a(-b)c\\ &=(a-b+c)(a^2+b^2+c^2+bc-ca+ab),\end{aligned}$$

$-b$ taking the place of b in formula (1).

Example 2.

$$\begin{aligned}x^3-8y^3-27-18xy &= x^3+(-2y)^3+(-3)^3-3x(-2y)(-3)\\ &=(x-2y-3)(x^2+4y^2+9-6y+3x+2xy).\end{aligned}$$

EXAMPLES XXVIII. a.

Resolve into factors :

1. x^4+16x^2+256.
2. $81a^4+9a^2b^2+b^4$.
3. $x^4+y^4-7x^2y^2$.
4. $m^4+n^4-18m^2n^2$.
5. $x^4-6x^2y^2+y^4$.
6. $4x^4+9y^4-93x^2y^2$.
7. $4m^4+9n^4-24m^2n^2$.
8. $9x^4+4y^4+11x^2y^2$.
9. $x^4-19x^2y^2+25y^4$.
10. $16a^4+b^4-28a^2b^2$.

11. $\dfrac{27}{a^3b^3} - 1.$ 12. $216a^3 - \dfrac{b^3}{8}.$ 13. $\dfrac{x^3}{125} + y^3.$

14. $\dfrac{m^3n^3}{729} - 1.$ 15. $\dfrac{a^3b^3}{125} + 1000.$ 16. $\dfrac{x^3}{512} - \dfrac{64}{x^3}.$

Resolve into two or more factors :

17. $x^2y + 3xy^2 - 3x^3 - y^3.$ 18. $4mn^2 - 20n^3 + 45mn^2 - 9m^3.$

19. $ab(x^2+1) + x(a^2+b^2).$ 20. $y^2z^2(x^4-1) + x^2(y^4-z^4).$

21. $a^3 + (a+b)ax + bx^2.$ 22. $pn(m^2+1) - m(p^2+n^2).$

23. $6bx(a^2+1) - a(4x^2+9b^2).$ 24. $(2a^2+3y^2)x + (2x^2+3a^2)y.$

25. $(2x^2-3a^2)y + (2a^2-3y^2)x.$

26. $a(a-1)x^2 + (2a^2-1)x + a(a+1).$

27. $3x^2 - (4a+2b)x + a^2 + 2ab.$

28. $2a^2x^2 - 2(3b-4c)(b-c)y^2 + abxy.$

29. $(a^2-3a+2)x^2 + (2a^2-4a+1)x + a(a-1).$

30. $a(a+1)x^2 + (a+b)xy - b(b-1)y^2.$

31. $b^3 + c^3 - 1 + 3bc.$ 32. $a^3 + 8c^3 + 1 - 6ac.$

33. $a^3 + b^3 + 8c^3 - 6abc.$ 34. $a^3 - 27b^3 + c^3 + 9abc.$

35. $a^3 - b^3 - c^3 - 3abc.$ 36. $8a^3 + 27b^3 + c^3 - 18abc.$

37. Resolve $x^8 + 81x^4 + 6561$ into three factors.

38. Resolve $(a^4 - 2a^2b^2 - b^4)^2 - 4a^4b^4$ into four factors.

39. Resolve $4(ab+cd)^2 - (a^2+b^2-c^2-d^2)^2$ into four factors.

40. Resolve $x^8 - \dfrac{1}{256}$ into four factors.

41. Resolve $x^{16} - y^{16}$ into five factors.

42. Resolve $x^{18} - y^{18}$ into six factors.

Resolve into four factors :

43. $\dfrac{a^3}{x^2} - 8x - a^3 + 8x^3.$ 44. $x^9 + x^3y^6 - 8x^6y^3 - 8y^9.$

45. $x^9 + x^6 + 64x^3 + 64.$ 46. $4a - 9b + \dfrac{4b^3}{a^2} - \dfrac{9a^3}{b^2}.$

47. $\dfrac{xy^3}{72} - \dfrac{x^3y^5}{32} - \dfrac{1}{9x^2} + \dfrac{y^2}{4}.$ 48. $x^6 - 25x^2 + 6\frac{1}{4} - \dfrac{1}{4}x^4.$

Resolve into five factors :

49. $x^7 + x^4 - 16x^3 - 16.$ 50. $16x^7 - 81x^3 - 16x^4 + 81.$

214. The actual processes of multiplication and division can often be partially or wholly avoided by a skilful use of factors.

It should be observed that the formulæ which the student has seen exemplified in the preceding pages are just as useful in their converse as in their direct application. Thus the formula for resolving into factors the difference of two squares is equally useful as enabling us to write down at once the product of the sum and the difference of two quantities.

Example 1. Multiply $2a+3b-c$ by $2a-3b+c$.

These expressions may be arranged thus:

$$2a+(3b-c) \text{ and } 2a-(3b-c).$$

Hence the product $=\{2a+(3b-c)\}\{2a-(3b-c)\}$

$$=(2a)^2-(3b-c)^2 \qquad \text{[Art. 133.]}$$
$$=4a^2-(9b^2-6bc+c^2)$$
$$=4a^2-9b^2+6bc-c^2.$$

Example 2. Multiply $(a^2+a+1)x-a-1$ by $(a-1)x-a^2+a-1$.

The product $=\{(a^2+a+1)x-(a+1)\}\{(a-1)x-(a^2-a+1)\}$

$$=(a^3-1)x^2-\{(a^4+a^2+1)+(a^2-1)\}x+(a^3+1)$$
$$=(a^3-1)x^2-(a^4+2a^2)x+a^3+1$$
$$=(a^3-1)x^2-a^2(a^2+2)x+a^3+1.$$

Note. The product of a^2+a+1 and a^2-a+1 is a^4+a^2+1 and should be written down without actual multiplication.

Example 3. Multiply $(3+x-2x^2)^2-(3-x+2x^2)^2$ ....(1),

by $(3+x+2x^2)^2-(3-x-2x^2)^2$ ....(2).

The expression (1)

$$=(3+x-2x^2+3-x+2x^2)(3+x-2x^2-3+x-2x^2)$$
$$=6(2x-4x^2)$$
$$=12x(1-2x).$$

The expression (2)

$$=(3+x+2x^2+3-x-2x^2)(3+x+2x^2-3+x+2x^2)$$
$$=6(2x+4x^2)$$
$$=12x(1+2x).$$

Therefore the product $=12x(1-2x)\times 12x(1+2x)$

$$=144x^2(1-4x^2).$$

Example 4. Divide the product of $2x^2+x-6$, and $6x^2-5x+1$ by $3x^2+5x-2$.

Denoting the division by means of a fraction, the required quotient

$$=\frac{(2x^2+x-6)(6x^2-5x+1)}{3x^2+5x-2}$$

$$=\frac{(2x-3)(x+2)(3x-1)(2x-1)}{(3x-1)(x+2)}$$

$$=(2x-3)(2x-1).$$

Example 5. Shew that $(2x+3y-z)^3+(3x+7y+z)^3$ is divisible by $5(x+2y)$.

The given expression is of the form A^3+B^3, and therefore has a divisor of the form $A+B$.

Therefore $(2x+3y-z)^3+(3x+7y+z)^3$

is divisible by $(2x+3y-z)+(3x+7y+z)$,

that is, by $5x+10y$,

or by $5(x+2y)$.

Example 6. Find the quotient when $a^3+8-5b(25b^2-6a)$ is divided by $a-5b+2$.

The expression $=a^3+8-125b^3+30ab$

$$=a^3+(-5b)^3+(2)^3-3\,.\,a(-5b)(2)$$

$$=(a-5b+2)(a^2+25b^2+4+10b-2a+5ab).$$

[Art. 213.]

$\therefore$ the quotient is $a^2+25b^2+4+10b-2a+5ab$.

Example 7. If $x+y=a$, and $x-y=b$ shew that

$$4(x^4-6x^2y^2+y^4)=6a^2b^2-a^4-b^4.$$

$$x^4-6x^2y^2+y^4=(x^4-2x^2y^2+y^4)-4x^2y^2$$

$$=(x^2-y^2)^2-\frac{1}{4}(4xy)^2$$

$$=\{(x+y)(x-y)\}^2-\frac{1}{4}\{(x+y)^2-(x-y)^2\}^2$$

$$=(ab)^2-\frac{1}{4}(a^2-b^2)^2;$$

$$\therefore\quad 4(x^4-6x^2y^2+y^4)=4a^2b^2-(a^2-b^2)^2$$

$$=6a^2b^2-a^4-b^4.$$

EXAMPLES XXVIII. b.

Find the product of

1. $2x-7y+3z$ and $2x+7y-3z$.
2. $3x^2-4xy+7y^2$ and $3x^2+4xy+7y^2$.
3. $5x^2+5xy-9y^2$ and $5x^2-5xy-9y^2$.
4. $7x^2-8xy+3y^2$ and $7x^2+8xy-3y^2$.
5. $x^3+2x^2y+2xy^2+y^3$ and $x^3-2x^2y+2xy^2-y^3$.
6. $(x+y)^2+2(x+y)+4$ and $(x+y)^2-2(x+y)+4$.
7. $(1+x+2x^2)^2-(1-x-2x^2)^2$ and $(1+x-2x^2)^2-(1-x+2x^2)^2$.
8. $(a^2+3a-1)^2-(a^2-3a-1)^2$ and $(a^2+a+1)^2-(a^2-a+1)^2$.
9. x^3-4x^2+8x-8 and x^3+4x^2+8x+8.
10. $x^3-6ax^2+18a^2x-27a^3$ and $x^3+6ax^2+18a^2x+27a^3$.
11. $x-a-\dfrac{x^2}{a}-\dfrac{a^2}{x}$ and $x+a+\dfrac{x^2}{a}-\dfrac{a^2}{x}$.
12. $(2x^2+3x+1)^2-(2x^2-3x-1)^2$ and $(x^2+6x-2)^2-(x^2-6x+2)^2$.

Find the continued product of

13. x^2+ax+a^2, x^2-ax+a^2, $x^4-a^2x^2+a^4$.
14. $1-x+x^2$, $1+x+x^2$, $1-x^2+x^4$, $1-x^4+x^8$.
15. $(a-x)^3$, $(a+x)^3$, $(a^2+x^2)^3$.
16. $(1-x)^2$, $(1+x)^2$, $(1+x^2)^2$, $(1+x^4)^2$.
17. x^2+4x+3, x^2+x-2, x^2-5x+6.
18. x^2+2x-3, x^2-5x+6, x^2+3x+2.
19. $x+2$, x^2+2x+4, $x-2$, x^2-2x+4.
20. Multiply the square of $a+3b$ by $a^2-6ab+9b^2$.
21. Multiply $\frac{1}{2}(a-b)^2+\frac{1}{2}(b-c)^2+\frac{1}{2}(c-a)^2$ by $a+b+c$.
22. Divide $(4x+3y-2z)^2-(3x-2y+3z)^2$ by $x+5y-5z$.
23. Divide $x^8+16a^4x^4+256a^8$ by $x^2+2ax+4a^2$.
24. Divide $(3x+4y-2z)^2-(2x+3y-4z)^2$ by $x+y+2z$.
25. Divide the product of $x^2+7x+10$ and $x+3$ by x^2+5x+6.
26. Divide $2x(x^2-1)(x+2)$ by x^2+x-2.
27. Divide $5x(x-11)(x^2-x-156)$ by x^3+x^2-132x.

28. Divide x^6+19x^3-216 by $(x^2-3x+9)(x-2)$.

29. Divide $(5x^2-3x-6)^2-(2x^2-7x+9)^2$ by the product of $3x-5$ and $x+3$.

30. Divide a^9-b^9 by the product of a^2+ab+b^2 and $a^6+a^3b^3+b^6$.

31. Divide $(x^3-3x^2y)^2-(3xy^2-y^3)^2$ by $(x-y)^3$.

32. Divide $(x^2-yz)^3+8y^3z^3$ by x^2+yz.

33. Divide $18xy+1+27x^3-8y^3$ by $1+3x-2y$.

34. Divide $(2x^2+3x-1)^2-(x^2+4x+5)^2$ by the product of $3x+4$ and $x+2$.

35. Divide the product of $6a^2-23a+20$ and $22a^2-81a+14$ by $33a^2-50a+8$.

36. Divide the product of $x^2+(a-b)x-ab$ and $x^2-(a-b)x-ab$ by $x^2+(a+b)x+ab$.

37. Divide $a^3-8y^3-9x(3x^2+2ay)$ by $a-3x-2y$.

38. Divide $27-8x^3-64y^3-72xy$ by $3-2(x+2y)$.

39. Shew that $(2x-3y+1)^3-(1-3x+2y)^3$ is divisible by $5(x-y)$.

40. Shew that the square of $x+1$ exactly divides
$$(x^3+x^2+4)^3-(x^3-2x+3)^3.$$

41. Shew that $2b+2d$ is a factor of the expression
$$(a+b+c+d)^3-(a-b+c-d)^3.$$

42. Shew that $(3x^2-7x+2)^3-(x^2-8x+8)^3$ is divisible by $2x-3$ and by $x+2$.

43. Shew that $(7x^2+3x-3)^3+(5x^2-4x-3)^3$ is divisible by $4x-3$ and by $3x+2$.

44. Shew that the sum of the cubes of $2x^2-5x-9$ and x^2+6x-5 is divisible by the product of $3x+7$ and $x-2$.

45. If $x+y=m$ and $x-y=n$, express x^3+y^3 in terms of m and n.

46. If $x+y=m$ and $x-y=n$, shew that
$$16(x^4-7x^2y^2+y^4)=(5m^2-n^2)(5n^2-m^2).$$

47. Find the value of $x^4+x^2y^2+y^4$ when $x+y=2a$, $x-y=2b$.

48. If $x+y=2a$ and $x-y=2b$ prove that
$$x^4-23x^2y^2+y^4=(7a^2-3b^2)(7b^2-3a^2).$$

49. Find the value of $x^4-47x^2y^2+y^4$ in terms of p and q when $x+y=p$ and $x-y=q$.

50. Find the value of $x^4-2x^3y+2xy^3-y^4$ when $x=a+b$ and $y=a-b$.

CHAPTER XXIX.

Miscellaneous Theorems and Examples.

215. Examples upon the simple rules, *e.g.* Division, Highest Common Factor, Evolution, etc., frequently occur which cannot be neatly and concisely worked without a ready use of factors and compound expressions. These we have hitherto excluded as unsuitable for the student until he has gained confidence and power by practice. We propose in the present chapter to bring together a miscellaneous collection of examples, for the most part not new in principle, but requiring some skill for their solution. The chapter will be found useful as a revision of the earlier chapters.

Example. Divide

$$ax^4-(ap-b)x^3+(aq-bp-c)x^2+(bq+cp)x-cq \text{ by } ax^2+bx-c.$$

$$
\begin{array}{r|l|l}
ax^2+bx-c & ax^4-(ap-b)x^3+(aq-bp-c)x^2+(bq+cp)x-cq & x^2-px+q \\
 & \underline{ax^4+bx^3-cx^2\qquad\qquad\qquad\qquad\qquad} & \\
 & \qquad -apx^3+(aq-bp)x^2+(bq+cp)x & \\
 & \qquad \underline{-apx^3-bpx^2 \qquad\quad +cpx} & \\
 & \qquad\qquad\qquad aqx^2 \qquad +bqx-cq & \\
 & \qquad\qquad\qquad \underline{aqx^2 \qquad +bqx-cq} &
\end{array}
$$

Note. When the coefficients in divisor or dividend are compound quantities it is best to retain them in brackets throughout the work.

216. In the process of finding the highest common factor, by the rules explained in Chap. XVIII., every remainder that occurs in the course of the work contains the factor we are seeking. Hence when any one of the remainders admits of being resolved into factors, we may often shorten the work.

Example 1. Find the H.C.F. of $2x^3-(4a-3c)x^2+6(b-ac)x+9bc$ and $2x^3+(2a+3c)x^2+(3ac-4b)x-6bc$.

$$
\begin{array}{r|l|l}
2x^3-(4a-3c)x^2+6(b-ac)x+9bc & 2x^3+(2a+3c)x^2+(3ac-4b)x-6bc & 1 \\
 & \underline{2x^3-(4a-3c)x^2+(6b-6ac)x+9bc} & \\
 & \qquad\qquad 6ax^2+(9ac-10b)x-15bc &
\end{array}
$$

$$
\begin{aligned}
\text{Now the remainder} &= 6ax^2+9acx-10bx-15bc \\
&= 3ax(2x+3c)-5b(2x+3c) \\
&= (2x+3c)(3ax-5b).
\end{aligned}
$$

Of these factors, $3ax-5b$ may clearly be rejected; therefore if there is a common factor it must be $2x+3c$. And by division, or by the method explained in Art. 152, we find that $2x+3c$ is a factor of each expression.

Hence the H.C.F. is $2x+3c$.

Example 2. Find the H.C.F. of $(a^2-2a)x^2+2(2a-1)x-a^2+1$ and $(a^2-a-2)x^2+(4a+1)x-a^2-a$.

Each of these expressions can be resolved into factors as explained in Art. 212, Ex. 4. Thus

$$(a^2-2a)x^2+2(2a-1)x-a^2+1=a(a-2)x^2+2(2a-1)x-(a+1)(a-1)$$
$$=\{(a-2)x+(a+1)\}\{ax-(a-1)\}.$$
$$(a^2-a-2)x^2+(4a+1)x-a^2-a=(a-2)(a+1)x^2+(4a+1)x-a(a+1)$$
$$=\{(a-2)x+(a+1)\}\{(a+1)x-a\}.$$

Hence the H.C.F. is $(a-2)x+a+1$.

EXAMPLES XXIX. a.

Divide

1. $x^3+(a+b+c)x^2+(bc+ca+ab)x+abc$ by $x^2+(a+b)x+ab$.
2. $x^4-(5+a)x^3+(4+5a+b)x^2-(4a+5b)x+4b$ by x^2-5x+4.
3. $x^3-(a-b)x^2-(ab+2b^2)x+2ab^2$ by $x-b$.
4. $x^3-(p^2+3q^2)x+2p^2q-2q^3$ by $x+p+q$.
5. $x^3-(3mn+n^2)x+m(m^2-n^2)$ by $x+m+n$.
6. $a(a-1)x^2+(2a^2-1)x+a(a+1)$ by $(a-1)x+a$.
7. $x^4+(a+b)x^3+(a^2+ab+b^2)x^2+(a^3+b^3)x+a^2b^2$ by x^2+ax+b^2.
8. $2l^2x^2-2(3m-4n)(m-n)y^2+lmxy$ by $lx+2(m-n)y$.
9. $(a^2+a-2)x^2-(2a+1)xy-(a^2+a)y^2$ by $(a-1)x-ay$.
10. $x^3-(a-b-2)x^2-(ab+2a-2b)x-2ab$ by $(x-a)(x+2)$.
11. $(x+1)^8+4(x+1)^6+6(x+1)^4+4(x+1)^2+1$ by x^2+2x+2.
12. $(m+1)(bx+an)b^2x^2-(n+1)(mbx+a)a^2$ by $bx-a$.

Find the H.C.F. of

13. $(m^2-3m+2)x^2+(2m^2-4m+1)x+m(m-1)$ and $m(m-1)x^2+(2m^2-1)x+m(m+1)$.
14. $mpx^3+(mq-np)x^2-(mr+nq)x+nr$ and $max^3-(mc+na)x^2-(mb-nc)x+nb$.
15. $2ap^3+(3a-2b)p^2q+(a-3b)pq^2-bq^3$ and $3ap^3-(a+3b)p^2q+(2a+b)pq^2-2bq^3$.
16. $acx^3+(bc+ad)x^2+(bd+ac)x+bc$ and $2acx^3+(2bc-ad)x^2-(3ac+bd)x-3bc$.

Find the H.C.F. of

17. $2a^2x^3-(4b+3)ax^2+2(3b-ac)x+3c$ and
$2a^2x^3+(2b-3)ax^2-(4ac+3b)x+6c.$

18. $2ax^3+(4a^2-1)bx^2-(2ab^2+3c)x-6abc$ and
$ax^3-(3-2a^2)bx^2+(2c-6ab^2)x+4abc.$

Find the L.C.M. of

19. $x^4-px^3+(q-1)x^2+px-q$ and $x^4-qx^3+(p-1)x^2+qx-p.$

20. $p(p+1)x^2+x-p(p-1)$ and $p(p+2)x^2+2x-p^2+1.$

21. $(a^2-5a+6)x^2+2(a-1)x-a(a+1)$ and
$a(a-3)x^2+12x-(a+1)(a+4).$

217. We add some miscellaneous questions in Evolution.

The *fourth* root of an expression is obtained by extracting the square root of the square root of the expression.

Similarly by successive applications of the rule for finding the square root, we may find the *eighth*, *sixteenth* ... root. The *sixth* root of an expression is found by taking the cube root of the square root, or the square root of the cube root.

Similarly by combining the two processes for extraction of cube and square roots, certain other higher roots may be obtained.

Example 1. Find the fourth root of
$$81x^4-216x^3y+216x^2y^2-96xy^3+16y^4.$$

Extracting the square root by the rule we obtain $9x^2-12xy+4y^2$; and *by inspection*, the square root of this is $3x-2y$, which is the required fourth root.

Example 2. Find the sixth root of
$$\left(x^3-\frac{1}{x^3}\right)^2-6\left(x-\frac{1}{x}\right)\left(x^3-\frac{1}{x^3}\right)+9\left(x-\frac{1}{x}\right)^2.$$

By inspection, the square root of this is
$$\left(x^3-\frac{1}{x^3}\right)-3\left(x-\frac{1}{x}\right),$$

which may be written $\qquad x^3-3x+\frac{3}{x}-\frac{1}{x^3}$;

and the cube root of this is $\qquad x-\frac{1}{x}$,

which is the required sixth root.

218. In Chap. VI. we have given examples of inexact division. In a similar manner when an expression is not an exact square

or cube, we may perform the process of evolution, and obtain as many terms of the root as we please.

Example. To find four terms of the square root of $1+2x-2x^2$.

$$\begin{array}{ll}
 & 1+2x-2x^2\left(1+x-\frac{3}{2}x^2+\frac{3}{2}x^3\right. \\
 & 1 \\
\hline
2+x & \quad 2x-2x^2 \\
 & \quad 2x+x^2 \\
\hline
2+2x-\frac{3}{2}x^2 & \qquad -3x^2 \\
 & \qquad -3x^2-3x^3+\frac{9}{4}x^4 \\
\hline
2+2x-3x^2+\frac{3}{2}x^3 & \qquad\qquad 3x^3-\frac{9}{4}x^4 \\
 & \qquad\qquad 3x^3+3x^4-\frac{9}{2}x^5+\frac{9}{4}x^6 \\
\hline
 & \qquad\qquad -\frac{21}{4}x^4+\frac{9}{2}x^5-\frac{9}{4}x^6.
\end{array}$$

Thus the required result is $1+x-\frac{3}{2}x^2+\frac{3}{2}x^3$.

***219.** In Art. 124 we pointed out the similarity between the arithmetical and algebraical methods of extracting square and cube roots. We shall now shew that in extracting either the square or the cube root of any number, when a certain number of figures have been obtained by the common rule, that number may be nearly doubled by ordinary division.

***220.** *If the square root of a number consists of* $2n+1$ *figures, when the first* $n+1$ *of these have been obtained by the ordinary method, the remaining* n *may be obtained by division.*

Let N denote the given number; a the part of the square root already found, that is the first $n+1$ figures found by the common rule, with n ciphers annexed; x the remaining part of the root.

Then
$$\sqrt{N}=a+x;$$
$$\therefore\ N=a^2+2ax+x^2;$$
$$\therefore\ \frac{N-a^2}{2a}=x+\frac{x^2}{2a}\ \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Now $N-a^2$ is the remainder after $n+1$ figures of the root, represented by a, have been found; and $2a$ is the divisor at the

same stage of the work. We see from (1) that $N - a^2$ divided by $2a$ gives x, the rest of the quotient required, increased by $\frac{x^2}{2a}$. We shall shew that $\frac{x^2}{2a}$ is a *proper fraction*, so that by neglecting the remainder arising from the division, we obtain x, the rest of the root.

For x contains n figures, and therefore x^2 contains $2n$ figures at most; also a is a number of $2n+1$ figures (the last n of which are ciphers) and thus $2a$ contains $2n+1$ figures at least; and therefore $\frac{x^2}{2a}$ is a proper fraction.

From the above investigation, by putting $n=1$, we see that *two* at least of the figures of a square root must have been obtained in order that the method of division, which is employed to obtain the next figure of the square root, may give that figure correctly.

Example. Find the square root of 290 to five places of decimals.

$$
\begin{array}{r|l}
 & \dot{2}9\dot{0}\,(\,17{\cdot}02 \\
 & 1 \\
\hline
27 & 190 \\
 & 189 \\
\hline
3402 & 10000 \\
 & 6804 \\
\hline
 & 3196
\end{array}
$$

Here we have obtained four figures in the square root by the ordinary method. Three more may be obtained by division only, using 2×1702, that is 3404, for divisor, and 3196 as remainder. Thus

$$
\begin{array}{l}
3404\,)\,31960\,(\,938 \\
30636 \\
\hline
13240 \\
10212 \\
\hline
30280 \\
27232 \\
\hline
3048
\end{array}
$$

And therefore to five places of decimals $\sqrt{290} = 17{\cdot}02938$.

When the divisor consists of several digits, the method of contracted division may be employed with advantage.

Again, it may be noticed that in obtaining the second figure of the root, the division of 190 by 20 gives 9 for the next figure; this is too great, and the figure 7 has to be obtained tentatively. This is one of the modifications of the algebraical rule to which we referred in Art. 124.

***221.** *If the cube root of a number consists of* $2n+2$ *figures, when the first* $n+2$ *of these have been obtained by the ordinary method, the remaining* n *may be obtained by division.*

Let N denote the given number ; a the part of the cube root already found, that is the first $n+2$ figures found by the common rule, with n ciphers annexed ; x the remaining part of the root.

Then $$\sqrt[3]{N}=a+x;$$

$$\therefore \quad N=a^3+3a^2x+3ax^2+x^3;$$

$$\therefore \quad \frac{N-a^3}{3a^2}=x+\frac{x^2}{a}+\frac{x^3}{3a^2} \quad \text{.............................}(1)$$

Now $N-a^3$ is the remainder after $n+2$ figures of the root, represented by a, have been found ; and $3a^2$ is the divisor at the same stage of the work. We see from (1) that $N-a^3$ divided by $3a^2$ gives x, the rest of the quotient required, increased by $\frac{x^2}{a}+\frac{x^3}{3a^2}$. We shall shew that this expression is a *proper fraction*, so that by neglecting the remainder arising from the division, we obtain x, the rest of the root.

By supposition, x is $<10^n$, and a is $>10^{2n+1}$;

$$\therefore \quad \frac{x^2}{a} \text{ is } < \frac{10^{2n}}{10^{2n+1}}; \text{ that is, } < \frac{1}{10};$$

and $$\frac{x^3}{3a^2} \text{ is } < \frac{10^{3n}}{3\times 10^{4n+2}}; \text{ that is, } < \frac{1}{3\times 10^{n+1}};$$

hence $$\frac{x^2}{a}+\frac{x^3}{3a^2} \text{ is } < \frac{1}{10}+\frac{1}{3\times 10^{n+1}},$$

and is therefore a proper fraction.

EXAMPLES XXIX. b.

Find the fourth roots of the following expressions :

1. $x^4-28x^3+294x^2-1372x+2401$.

2. $16-\frac{32}{m}+\frac{24}{m^2}-\frac{8}{m^3}+\frac{1}{m^4}$.

3. $a^4+8a^3x+16x^4+32ax^3+24a^2x^2$.

4. $1+4x+2x^2-8x^3-5x^4+8x^5+2x^6-4x^7+x^8$.

5. $1+8x+20x^2+8x^3-26x^4-8x^5+20x^6-8x^7+x^8$.

Find the sixth roots of the following expressions:

6. $1+6x+15x^2+20x^3+15x^4+6x^5+x^6$.

7. $x^6-12ax^5+240a^4x^2-192a^5x+60a^2x^4-160a^3x^3+64a^6$.

8. $a^6-18a^5x+135a^4x^2-540a^3x^3+1215a^2x^4-1458ax^5+729x^6$.

Find the eighth roots of the following expressions:

9. $x^8-8x^7y+28x^6y^2-56x^5y^3+70x^4y^4-56x^3y^5+28x^2y^6-8xy^7+y^8$.

10. $\{x^4+2(p-1)x^3+(p^2-2p-1)x^2-2(p-1)x+1\}^4$.

Find to four terms the square root of

11. $1+x$. 12. $1-2x$. 13. $4+2x$. 14. $1-x-x^2$.
15. a^2-x. 16. x^2+a^2. 17. a^4-3x^2. 18. $9a^2+12ax$.

Find to three terms the cube root of

19. x^3-a^3. 20. $8+x$. 21. $\frac{1}{a^3}+9x$.
22. $1-6x+21x^2$. 23. $27x^6-27x^5-18x^4$. 24. $64-48x+9x^2$.

Identities and Transformations.

***222.** Definition. An **identity** is an algebraical statement which is true for all values of the letters involved in it.

Examples. $a^3+b^3=(a+b)(a^2-ab+b^2)$.
$$x^3+y^3+z^3-3xyz=(x+y+z)(x^2+y^2+z^2-yz-zx-xy).$$

***223.** An identity asserts that two expressions are always equal; and the proof of this equality is called "proving the identity." The method of procedure is to choose one of the expressions given, and to shew by successive transformations that it can be made to assume the form of the other.

Example 1. To prove that
$$bc(b-c)+ca(c-a)+ab(a-b)=-(b-c)(c-a)(a-b).$$
The first side $=bc(b-c)+c^2a-ca^2+a^2b-ab^2$
$$\begin{aligned}&=bc(b-c)+a^2(b-c)-a(b^2-c^2)\\&=(b-c)\{bc+a^2-a(b+c)\}\\&=(b-c)\{bc+a^2-ab-ac\}\\&=(b-c)\{a(a-b)-c(a-b)\}\\&=(b-c)(a-b)(a-c)\\&=-(b-c)(c-a)(a-b),\end{aligned}$$
changing the signs of the factor $a-c$, so as to preserve cyclic order.
[Compare Art. 229, Example 3.]

The expression on the left-hand side can be readily put in the following forms:

$$a^2(b-c)+b^2(c-a)+c^2(a-b);$$
$$-\{a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)\}.$$

Hence we have the following results:

$$bc(b-c)+ca(c-a)+ab(a-b)=-(b-c)(c-a)(a-b);$$
$$a^2(b-c)+b^2(c-a)+c^2(a-b)=-(b-c)(c-a)(a-b);$$
$$a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)=(b-c)(c-a)(a-b).$$

These identities are of such frequent occurrence that they should be carefully noticed and remembered.

Example 2. If $2s=a+b+c$ prove that

$$\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c}-\frac{1}{s}=\frac{abc}{s(s-a)(s-b)(s-c)}.$$

$$\begin{aligned}\text{The first side}&=\left(\frac{1}{s-a}+\frac{1}{s-b}\right)+\left(\frac{1}{s-c}-\frac{1}{s}\right)\\
&=\frac{s-b+s-a}{(s-a)(s-b)}+\frac{s-s+c}{s(s-c)}\\
&=\frac{2s-a-b}{(s-a)(s-b)}+\frac{c}{s(s-c)}\\
&=\frac{c}{(s-a)(s-b)}+\frac{c}{s(s-c)}\\
&=c\left\{\frac{s(s-c)+(s-a)(s-b)}{s(s-a)(s-b)(s-c)}\right\}\\
&=\frac{c\{s^2-cs+s^2-as-bs+ab\}}{s(s-a)(s-b)(s-c)}\\
&=\frac{c\{2s^2-s(a+b+c)+ab\}}{s(s-a)(s-b)(s-c)}\\
&=\frac{abc}{s(s-a)(s-b)(s-c)},\ \text{for } s(a+b+c)=s\,.\,2s=2s^2.\end{aligned}$$

Note. Here $2s$ is a convenient abbreviation of $a+b+c$; and the reduction is much simplified by working in terms of s instead of substituting its value at once. In examples of this kind, as a rule, the student should avoid substituting as long as the work can be carried on in terms of the symbol of abbreviation.

Example 3. If $x^2+u^2=2(xy+yz+zu-y^2-z^2)$

prove that $\qquad x=y=z=u.$

By transposing, we have

$$x^2-2xy+y^2+y^2-2yz+z^2+z^2-2zu+u^2=0,$$

or $\qquad (x-y)^2+(y-z)^2+(z-u)^2=0.$

Now since the square of any quantity is always positive, each of the expressions $(x-y)^2$, $(y-z)^2$, $(z-u)^2$ is positive. Hence their sum cannot be zero unless each of them be separately equal to zero.

$$\therefore \quad x-y=0, \quad y-z=0, \quad z-u=0;$$

or

$$x=y=z=u.$$

Note. The student should be careful to notice the difference between the conclusions to be drawn from the two statements

$$(x-a)^2+(y-b)^2=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

and

$$(x-a)(y-b)=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

From (1) we infer that *both* $x-a=0$ and $y-b=0$ *simultaneously*, while from (2) we infer that *either* $x-a=0$ *or* $y-b=0$.

*EXAMPLES XXIX. c.

Prove the following identities:

1. $b(x^3+a^3)+ax(x^2-a^2)+a^3(x+a)=(a+b)(x+a)(x^2-ax+a^2)$.

2. $(ax+by)^2+(ay-bx)^2+c^2x^2+c^2y^2=(x^2+y^2)(a^2+b^2+c^2)$.

3. $(x+y)^3+3(x+y)^2z+3(x+y)z^2+z^3$
$$=(x+z)^3+3(x+z)^2y+3(x+z)y^2+y^3.$$

4. $(a+b+c)(ab+bc+ca)-abc=(a+b)(b+c)(c+a)$.

5. $(a+b+c)^2-a(b+c-a)-b(a+c-b)-c(a+b-c)=2(a^2+b^2+c^2)$.

6. $(x-y)^3+(x+y)^3+3(x-y)^2(x+y)+3(x+y)^2(x-y)=8x^3$.

7. $x^2(y-z)+y^2(z-x)+z^2(x-y)+(y-z)(z-x)(x-y)=0$.

8. $a^3(b-c)+b^3(c-a)+c^3(a-b)=-(b-c)(c-a)(a-b)(a+b+c)$.

9. If $x+y+z=0$, prove that $x^3+y^3+z^3=3xyz$.

10. Prove that $(b-c)^3+(c-a)^3+(a-b)^3=3(b-c)(c-a)(a-b)$.

If $2s=a+b+c$, shew that

11. $(s-a)^2+(s-b)^2+(s-c)^2+s^2=a^2+b^2+c^2$.

12. $(s-a)^3+(s-b)^3+(s-c)^3+3abc=s^3$.

13. $16s(s-a)(s-b)(s-c)=2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4$.

14. $2(s-a)(s-b)(s-c)+a(s-b)(s-c)+b(s-c)(s-a)$
$$+c(s-a)(s-b)=abc.$$

If $a+b+c=0$, shew that

15. $(2a-b)^3+(2b-c)^3+(2c-a)^3=3(2a-b)(2b-c)(2c-a)$.

16. $\dfrac{a^2}{2a^2+bc}+\dfrac{b^2}{2b^2+ca}+\dfrac{c^2}{2c^2+ab}=1$.

17. Prove that
$$(x+y+z)^3+(x+y-z)^3+(x-y+z)^3+(x-y-z)^3=4x(x^2+3y^2+3z^2).$$

18. If $a+b+c=s$, prove that

$$(s-3a)^3+(s-3b)^3+(s-3c)^3-3(s-3a)(s-3b)(s-3c)=0.$$

19. If $X=b+c-2a$, $Y=c+a-2b$, $Z=a+b-2c$, find the value of $X^3+Y^3+Z^3-3XYZ$.

20. Find the value of $a(a^2+bc)+b(b^2+ac)-c(c^2-ab)$ when $a=\cdot 7$, $b=\cdot 08$, $c=\cdot 78$.

21. Prove that $(a-b)^2+(b-c)^2+(c-a)^2$

$$=2(c-b)(c-a)+2(b-a)(b-c)+2(a-b)(a-c).$$

22. Prove that $a^2(b^3-c^3)+b^2(c^3-a^3)+c^2(a^3-b^3)$

$$\begin{aligned}&=(a-b)(b-c)(c-a)(ab+bc+ca)\\&=a^2(b-c)^3+b^2(c-a)^3+c^2(a-b)^3\\&=-[a^2b^2(a-b)+b^2c^2(b-c)+c^2a^2(c-a)].\end{aligned}$$

23. If $(a+b)^2+(b+c)^2+(c+d)^2=4(ab+bc+cd)$, prove that

$$a=b=c=d.$$

24. If $x=a+d$, $y=b+d$, $z=c+d$, prove that

$$x^2+y^2+z^2-yz-zx-xy=a^2+b^2+c^2-bc-ca-ab.$$

25. If $a+b+c=0$, prove that

$$\frac{1}{b^2+c^2-a^2}+\frac{1}{c^2+a^2-b^2}+\frac{1}{a^2+b^2-c^2}=0.$$

26. If $a+b+c=0$, simplify

$$\frac{b+c}{bc}(b^2+c^2-a^2)+\frac{c+a}{ca}(c^2+a^2-b^2)+\frac{a+b}{ab}(a^2+b^2-c^2).$$

27. Prove that the equation

$$(x-a)^2+(y-b)^2+(a^2+b^2-1)(x^2+y^2-1)=0,$$

is equivalent to the equation

$$(ax+by-1)^2+(bx-ay)^2=0\,;$$

hence shew that the only possible values of x and y are

$$\frac{a}{a^2+b^2},\quad \frac{b}{a^2+b^2}.$$

28. If $2(x^2+a^2-ax)(y^2+b^2-by)=x^2y^2+a^2b^2$, shew that

$$(x-a)^2(y-b)^2+(bx-ay)^2=0,$$

and therefore that $x=a$, $y=b$ are the only possible solutions.

***224.** We shall now give some further examples of fractions to illustrate the advantage of arranging expressions with regard to cyclic order. [Art. 172.]

Example. Find the value of

$$\frac{a}{(a-b)(a-c)(x-a)}+\frac{b}{(b-c)(b-a)(x-b)}+\frac{c}{(c-a)(c-b)(x-c)}.$$

Changing the sign of one factor in each denominator, so as to preserve cyclic order, we get for the lowest common denominator,

$$(a-b)(b-c)(c-a)(x-a)(x-b)(x-c).$$

The whole expression has for its numerator

$$-[a(b-c)(x-b)(x-c)+\ldots\ldots+\ldots\ldots]$$

or $$-[a(b-c)\{x^2-(b+c)x+bc\}+\ldots\ldots+\ldots\ldots].$$

Arrange it according to powers of x; thus

$$\begin{aligned}\text{coefficient of } x^2 &= -\{a(b-c)+b(c-a)+c(a-b)\}\\ &=0;\\ \text{coefficient of } x &=\{a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)\}\\ &=(b-c)(c-a)(a-b); \qquad \text{[Art. 223.]}\end{aligned}$$

terms which do not contain x

$$\begin{aligned}&=-\{abc(b-c)+abc(c-a)+abc(a-b)\}\\ &=-abc\{b-c+c-a+a-b\}\\ &=0.\end{aligned}$$

Hence the expression $=\dfrac{(b-c)(c-a)(a-b)x}{(b-c)(c-a)(a-b)(x-a)(x-b)(x-c)}$

$$=\frac{x}{(x-a)(x-b)(x-c)}.$$

Note. In examples of this kind the work will be much facilitated if the student accustoms himself to readily writing down the following equivalents:

$$(b-c)+(c-a)+(a-b)=0.$$
$$a(b-c)+b(c-a)+c(a-b)=0.$$
$$a^2(b-c)+b^2(c-a)+c^2(a-b)=-(a-b)(b-c)(c-a).$$
$$bc(b-c)+ca(c-a)+ab(a-b)=-(a-b)(b-c)(c-a).$$
$$a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)=(a-b)(b-c)(c-a).$$

Some of the identities in XXIX. c. may also be remembered with advantage.

*EXAMPLES XXIX. d.

1. $\dfrac{a}{(a-b)(a-c)}+\dfrac{b}{(b-c)(b-a)}+\dfrac{c}{(c-a)(c-b)}.$

2. $\dfrac{bc}{(a-b)(a-c)}+\dfrac{ca}{(b-c)(b-a)}+\dfrac{ab}{(c-a)(c-b)}.$

3. $\dfrac{a^2}{(a-b)(a-c)}+\dfrac{b^2}{(b-c)(b-a)}+\dfrac{c^2}{(c-a)(c-b)}.$

4. $\dfrac{a^3}{(a-b)(a-c)}+\dfrac{b^3}{(b-c)(b-a)}+\dfrac{c^3}{(c-a)(c-b)}.$

5. $\dfrac{a(b+c)}{(a-b)(c-a)}+\dfrac{b(a+c)}{(a-b)(b-c)}+\dfrac{c(a+b)}{(c-a)(b-c)}.$

6. $\dfrac{1}{a(a-b)(a-c)}+\dfrac{1}{b(b-c)(b-a)}+\dfrac{1}{c(c-a)(c-b)}.$

7. $\dfrac{bc}{a(a^2-b^2)(a^2-c^2)}+\dfrac{ca}{b(b^2-c^2)(b^2-a^2)}+\dfrac{ab}{c(c^2-a^2)(c^2-b^2)}.$

8. $\dfrac{(x-b)(x-c)}{(a-b)(a-c)}+\dfrac{(x-c)(x-a)}{(b-c)(b-a)}+\dfrac{(x-a)(x-b)}{(c-a)(c-b)}.$

9. $\dfrac{bc(a+d)}{(a-b)(a-c)}+\dfrac{ca(b+d)}{(b-c)(b-a)}+\dfrac{ab(c+d)}{(c-a)(c-b)}.$

10. $\dfrac{1}{(a-b)(a-c)(x-a)}+\dfrac{1}{(b-c)(b-a)(x-b)}+\dfrac{1}{(c-a)(c-b)(x-c)}.$

11. $\dfrac{a^2}{(a-b)(a-c)(x+a)}+\dfrac{b^2}{(b-c)(b-a)(x+b)}+\dfrac{c^2}{(c-a)(c-b)(x+c)}$

12. $a^2\dfrac{(a+b)(a+c)}{(a-b)(a-c)}+b^2\dfrac{(b+c)(b+a)}{(b-c)(b-a)}+c^2\dfrac{(c+a)(c+b)}{(c-a)(c-b)}.$

13. $\dfrac{a^3(b-c)+b^3(c-a)+c^3(a-b)}{(b-c)^3+(c-a)^3+(a-b)^3}.$

14. $\dfrac{a^2(b-c)+b^2(c-a)+c^2(a-b)+2(a-b)(b-c)(c-a)}{(b-c)^3+(c-a)^3+(a-b)^3}.$

15. $\dfrac{a^3(b-c)+b^3(c-a)+c^3(a-b)}{a^2(b-c)+b^2(c-a)+c^2(a-b)}.$

16. $\dfrac{a^2(b-c)^3+b^2(c-a)^3+c^2(a-b)^3}{(a-b)(b-c)(c-a)}.$

17. $\dfrac{\frac{1}{a}(b-c)+\frac{1}{b}(c-a)+\frac{1}{c}(a-b)}{\frac{1}{a}\left(\frac{1}{b^2}-\frac{1}{c^2}\right)+\frac{1}{b}\left(\frac{1}{c^2}-\frac{1}{a^2}\right)+\frac{1}{c}\left(\frac{1}{a^2}-\frac{1}{b^2}\right)}.$

18. $\dfrac{a^2\left(\frac{1}{c^2}-\frac{1}{b^2}\right)+b^2\left(\frac{1}{a^2}-\frac{1}{c^2}\right)+c^2\left(\frac{1}{b^2}-\frac{1}{a^2}\right)}{\frac{1}{bc}\left(\frac{1}{c}-\frac{1}{b}\right)+\frac{1}{ca}\left(\frac{1}{a}-\frac{1}{c}\right)+\frac{1}{ab}\left(\frac{1}{b}-\frac{1}{a}\right)}.$

***225.** *To find when* x^3+px^2+qx+r .... (1),
is divisible by x^2+ax+b .... (2).

Divide (1) by (2) in the ordinary way; thus

$$\begin{array}{r|l|l} x^2+ax+b & x^3+px^2+qx+r & x+(p-a) \\ & x^3+ax^2+bx & \\ \hline & (p-a)x^2+\ \ (q-b)\ x+r & \\ & (p-a)x^2+a(p-a)\ x+b(p-a) & \\ \hline & \{(q-b)-a(p-a)\}x+r-b(p-a) & \end{array} \quad\text{.............(3)}$$

Now if the remainder is zero the division is exact. This is the case when

$$\{(q-b)-a(p-a)\}x+r-b(p-a)=0,$$

or

$$x=\frac{b(p-a)-r}{q-b-a(p-a)}$$

Hence when x has this value, (1) is divisible by (2)

But if in (3), $\quad q-b-a(p-a)=0,$
and also $\quad r-b(p-a)=0,$

the remainder is equal to zero *whatever value* x *may have.* Thus x^3+px^2+qx+b is divisible by x^2+ax+b for *all* values of x, provided that

$$q-b-a(p-a)=0,$$

and

$$r-b(p-a)=0.$$

***226.** *To find the condition that* x^2+px+q *may be a perfect square.*

Using the ordinary rule for square root, we have

$$\begin{array}{r|l} & x^2+px+q\ \Big(x+\frac{p}{2} \\ & x^2 \\ \hline 2x+\frac{p}{2} & px+q \\ & px+\frac{p^2}{4} \\ \hline & q-\frac{p^2}{4}. \end{array}$$

If therefore x^2+px+q be a perfect square, the remainder, $q-\frac{p^2}{4}$, must be zero.

Hence $q-\frac{p^2}{4}=0$, or $p^2=4q$, is the condition required.

***227.** *To prove that* $x^4+px^3+qx^2+rx+s$ *is a perfect square if* $\left(q-\frac{p^2}{4}\right)^2=4s$ and $r^2=p^2s$.

The square root must clearly be a trinomial expression of the form x^2+lx+m; if therefore we put

$$x^4+px^3+qx^2+rx+s=(x^2+lx+m)^2,$$

we have, on expanding the right-hand side

$$x^4+px^3+qx^2+rx+s=x^4+2lx^3+x^2(l^2+2m)+2lmx+m^2.$$

Since this is to be true for all values of x, *we may assume that the coefficients of the like powers of* x *are the same;* hence

$$2l=p, \qquad l^2+2m=q,$$
$$2lm=r, \qquad m^2=s.$$

From these equations, by eliminating the unknown quantities l and m, we shall obtain the necessary relations between p, q, r, and s.

Thus we have
$$q-\frac{p^2}{4}=2m=2\sqrt{s},$$
$$r=2lm=p\sqrt{s};$$
$$\therefore \quad \left(q-\frac{p^2}{4}\right)^2=4s \text{ and } r^2=p^2s.$$

Note. The method of Art. 226 might have been used here. Also the method of the present article may be used to establish the results of Arts. 225 and 226.

***228.** The proposition in the preceding article has been given to illustrate a useful method, which admits of very wide application. In the course of the proof we assume the truth of an important principle; namely,

If two rational integral expressions involving x *are identically equal, the coefficients of like powers of* x *in the two expressions are equal.*

[An expression is said to be *rational* when no term contains a square or other root, and it is said to be *integral with respect to* x when the powers of x are all positive integers.]

The demonstration of this principle belongs to a more advanced part of the subject, and could not be discussed completely here. [See *Higher Algebra*. Art. 311.]

The Remainder Theorem.

***229.** *If a rational integral algebraical expression*

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + p_3 x^{n-3} + \dots + p_{n-1} x + p_n$$

be divided by $x - a$, *the remainder will be*

$$a^n + p_1 a^{n-1} + p_2 a^{n-2} + p_3 a^{n-3} + \dots + p_{n-1} a + p_n.$$

Divide the given expression by $x - a$ till a remainder is obtained which does not involve x. Let Q be the quotient, and R the remainder ; then

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \dots + p_{n-1} x + p_n = Q(x-a) + R.$$

Since R does not contain x, it will remain unaltered whatever value we give to x.

Put $x = a$, then

$$a^n + p_1 a^{n-1} + p_2 a^{n-2} + \dots + p_{n-1} a + p_n = Q \times 0 + R.$$

$$\therefore \quad R = a^n + p_1 a^{n-1} + p_2 a^{n-2} + \dots + p_{n-1} a + p_n ;$$

which proves the proposition.

From this it appears that when an algebraical expression is divided by $x - a$, the remainder can be obtained at once by writing a in the place of x in the given expression.

Again, the remainder is zero when the given expression is exactly divisible by $x - a$; hence we deduce another important proposition, known as the **Factor Theorem.**

If a rational integral expression involving x *become equal to* 0 *when* a *is written for* x, *it will contain* $x - a$ *as a factor.*

Example 1. Resolve into factors $x^3 + 3x^2 - 13x - 15$.

By trial we find that this expression vanishes when $x = 3$; hence $x - 3$ is a factor.

$$\begin{aligned} \therefore \quad x^3 + 3x^2 - 13x - 15 &= x^2(x-3) + 6x(x-3) + 5(x-3) \\ &= (x-3)(x^2 + 6x + 5) \\ &= (x-3)(x+1)(x+5). \end{aligned}$$

Note. The only numerical values that need be substituted for x are the factors of the last term of the expression. Thus, in the present case, by making trial of -5, we should have detected the factor $x + 5$.

Example 2. The remainder when x^4-2x^3+x-7 is divided by $x+2$ is

$$(-2)^4-2(-2)^3+(-2)-7;$$

that is, $$16+16-2-7, \text{ or } 23.$$

Or the remainder may be found more shortly by substituting $x=-2$ in $[\{(x-2)x\}x+1]x-7$.

Example 3. Find the factors of $bc(b-c)+ca(c-a)+ab(a-b)$.

On trial, this expression vanishes when $b=c$; therefore $b-c$ is a factor. Similarly $c-a$, $a-b$ may be shewn to be factors.

$$\therefore\ bc(b-c)+ca(c-a)+ab(a-b)=M(b-c)(c-a)(a-b)\ldots\ldots(1);$$

and since the left-hand member of this identity is only of three dimensions in a, b, c, the factor M must be some numerical quantity independent of a, b, c; its value can therefore be found by giving particular values to a, b, c, or by equating the coefficients of like terms on each side.

Let $a=0$, $b=1$, $c=2$, then (1) becomes

$$2(-1)+0+0=M(-1)\times 2\times(-1);$$

whence $$M=-1.$$

$$\therefore\ bc(b-c)+ca(c-a)+ab(a-b)=-(b-c)(c-a)(a-b).$$

***230.** We shall now give general proofs of the statements made in Art. 55. We suppose n to be positive and integral.

I. *To prove that* x^n-y^n *is always divisible by* $x-y$.

By the remainder theorem when x^n-y^n is divided by $x-y$ the remainder is

$$y^n-y^n, \text{ or } 0,$$

that is, x^n-y^n is always divisible by $x-y$.

II. *To prove that* x^n+y^n *is divisible by* $x+y$ *when* n *is odd, but not when* n *is even.*

By the remainder theorem when x^n+y^n is divided by $x+y$ the remainder is

$$(-y)^n+y^n.$$

(1) if n is odd, $\quad (-y)^n+y^n=-y^n+y^n=0$;

(2) if n is even, $\quad (-y)^n+y^n=\ \ y^n+y^n=2y^n$;

hence there is a remainder when n is even, but none when n is odd; which proves the proposition.

In like manner it may be proved that x^n-y^n is divisible by $x+y$ when n is even; and x^n+y^n is *never* divisible by $x-y$.

By going through a few steps of the division, the form of the quotient in each case is easily determined. The results of the present article may be conveniently stated as follows:

(i) For all values of n,

$$x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \dots + y^{n-1}).$$

(ii) When n is odd,

$$x^n + y^n = (x+y)(x^{n-1} - x^{n-2}y + x^{n-3}y^2 - \dots + y^{n-1}).$$

(iii) When n is even,

$$x^n - y^n = (x+y)(x^{n-1} - x^{n-2}y + x^{n-3}y^2 - \dots - y^{n-1}).$$

*EXAMPLES XXIX. e.

Find the values of x which will make each of the following expressions a perfect square:

1. $x^4 + 6x^3 + 13x^2 + 13x - 1$.   2. $x^4 + 6x^3 + 11x^2 + 3x + 31$.

3. $x^4 - 2ax^3 + (a^2 + 2b)x^2 - 3abx + 2b^2$.

4. $4p^2x^4 - 4pqx^3 + (q^2 + 2p^2)x^2 - 5pqx + \dfrac{p^2}{2}$.

5. $\dfrac{a^2x^6}{9} - \dfrac{abx^4}{2} + \dfrac{2acx^3}{3} + \dfrac{9b^2x^2}{16} - \dfrac{5bcx}{2} + 6c^2$.

6. $x^4 + 2ax^3 + 3a^2x^2 + cx + d$.

7. Find the conditions that $x^4 - ax^3 + bx^2 - cx + 1$ may be a perfect square for all values of x.

Find the values of x which will make each of the following expressions a perfect cube:

8. $8x^3 - 36x^2 + 56x - 39$.   9. $\dfrac{x^6}{27} - \dfrac{a^2x^4}{3} + 4a^4x^2 - 28a^6$.

10. $m^3x^6 - 9m^2nx^4 + 39mn^2x^2 - 51n^3$.

11. Find the relation between b and c in order that

$$x^3 + 3ax^2 + bx + c$$

may be a perfect cube for all values of x.

12. Find the conditions that

$$x^6 + 3ax^5 + 3bx^4 + a(6b - 5a^2)x^3 + 3b(b - a^2)x^2 + 3cx + d$$

may be a perfect cube for all values of x.

13. What number must be added to $x^3 + 2x^2$ in order that the expression may be divisible by $x + 4$?

14. If $x + a$ be a common factor of $x^2 + px + q$ and $x^2 + lx + m$, shew that $a = \dfrac{m-q}{l-p}$.

Resolve into factors:

15. $x^3 - 6x^2 + 11x - 6$.
16. $x^3 - 5x^2 - 2x + 24$.
17. $x^3 + 9x^2 + 26x + 24$.
18. $x^3 - x^2 - 41x + 105$.
19. $x^3 - 39x + 70$.
20. $x^3 - 8x^2 - 31x - 22$.
21. $6x^3 + 7x^2 - x - 2$.
22. $6x^3 + x^2 - 19x + 6$.

Write down the quotient in the following cases:

23. $\dfrac{x^7 + y^7}{x + y}$.
24. $\dfrac{x^8 - y^8}{x + y}$.
25. $\dfrac{x^6 - y^6}{x - y}$.
26. $\dfrac{x^9 - y^9}{x - y}$.

Find the square root of

27. $x^4 + (2a - 4)x^3 + (a^2 - 2a + 4)x^2 + (2a^2 - 4a)x + a^2$.

28. $(a+1)^2x^4 + (2a^2 + 2a)x^3 + (3a^2 - 4a - 6)x^2 + (2a^2 - 6a)x + a^2 - 6a + 9$.

29. Find what values of m make $3mx^2 + (6m - 12)x + 8$ a perfect square.

30. If $4x^4 + 12x^3y + Px^2y^2 + 6xy^3 + y^4$ is a perfect square, find P.

Without actual division shew that

31. $32x^{10} - 33x^5 + 1$ is divisible by $x - 1$.
32. $3x^4 + 5x^3 - 13x^2 - 20x + 4$ .... $x^2 - 4$.
33. $x^4 + 4x^3 - 5x^2 - 36x - 36$ .... $x^2 - x - 6$.

Without actual division find the remainder when

34. $x^5 - 5x^2 + 5$ is divided by $x - 5$.
35. $x^3 - 7x^2a + 8xa^2 + 15a^3$ .... $x + 2a$.

36. If $ax^2 - bx + c$ and $dx^3 - bx + c$ have a common factor, then
$$a^3 - abd + cd^2 = 0.$$

37. If n be any positive integer, prove that $5^{2n} - 1$ is always divisible by 24.

38. Shew that $1 - x - x^n + x^{n+1}$ is exactly divisible by
$$1 - 2x + x^2.$$

39. If $x^3 + px + r$ and $3x^2 + p$ have a common factor, prove that
$$\frac{p^3}{27} + \frac{r^2}{4} = 0.$$

40. Shew that if $x^n + py^n + qz^n$ is exactly divisible by
$$x^2 - (ay + bz)x + abyz,$$
then
$$\frac{p}{a^n} + \frac{q}{b^n} + 1 = 0.$$

CHAPTER XXX.

THE THEORY OF INDICES.

231. HITHERTO all the definitions and rules with regard to indices have been based upon the supposition that they were positive integers; for instance

(1) $a^{14} = a \cdot a \cdot a \ldots$ to fourteen factors.

(2) $a^{14} \times a^3 = a^{14+3} = a^{17}$.

(3) $a^{14} \div a^3 = a^{14-3} = a^{11}$.

(4) $(a^{14})^3 = a^{14\times 3} = a^{42}$.

The object of the present chapter is twofold: first, to give *general* proofs which shall establish the laws of combination in the case of all positive integral indices; secondly, to explain how, in strict accordance with these laws, intelligible meanings may be given to symbols whose indices are fractional, zero, or negative.

We shall begin by proving, directly from the definition of a positive integral index, three important propositions.

232. DEFINITION. When m is a *positive integer*, a^m stands for the product of m factors each equal to a.

233. PROP. I. *To prove that* $a^m \times a^n = a^{m+n}$, *when* m *and* n *are positive integers.*

By definition, $a^m = a \cdot a \cdot a \ldots$ to m factors;

$a^n = a \cdot a \cdot a \ldots$ to n factors;

$$\therefore\ a^m \times a^n = (a \cdot a \cdot a \ldots \text{ to } m \text{ factors}) \times (a \cdot a \cdot a \ldots \text{ to } n \text{ factors})$$
$$= a \cdot a \cdot a \ldots \text{ to } m+n \text{ factors}$$
$$= a^{m+n}, \text{ by definition.}$$

COR. If p is also a positive integer, then

$$a^m \times a^n \times a^p = a^{m+n+p};$$

and so for any number of factors.

234. PROP. II. *To prove that* $a^m \div a^n = a^{m-n}$, *when* m *and* n *are positive integers, and* $m > n$.

$$a^m \div a^n = \frac{a^m}{a^n} = \frac{a.a.a \text{ ...... to } m \text{ factors}}{a.a.a \text{ ...... to } n \text{ factors}}$$
$$= a.a.a \text{ ...... to } m-n \text{ factors}$$
$$= a^{m-n}.$$

235. PROP. III. *To prove that* $(a^m)^n = a^{mn}$, *when* m *and* n *are positive integers.*

$$(a^m)^n = a^m.a^m.a^m \text{ ...... to } n \text{ factors}$$
$$= (a.a.a \text{ ... to } m \text{ factors})(a.a.a \text{ ... to } m \text{ factors}) \text{ ...}$$

the bracket being repeated n times,

$$= a.a.a \text{ ...... to } mn \text{ factors}$$
$$= a^{mn}.$$

236. These are the fundamental laws of combination of indices, and they are proved directly from a definition which is intelligible only on the supposition that the indices are *positive* and *integral.*

But it is found convenient to use fractional and negative indices, such as $a^{\frac{4}{5}}$, a^{-7}, or, more generally, $a^{\frac{p}{q}}$, a^{-n}; and these have at present no intelligible meaning. For it is plain that the definition of a^m, [Art. 232], upon which we based the three propositions just proved, is no longer applicable when m is *fractional*, or *negative.*

Now it is important that all indices, whether positive or negative, integral or fractional, should be governed by the same laws. We therefore determine meanings for symbols such as $a^{\frac{p}{q}}$, a^{-n}, in the following way: we assume that they conform to the fundamental law, $a^m \times a^n = a^{m+n}$, and accept the meaning to which this assumption leads us. It will be found that the symbols so interpreted will also obey the other laws enunciated in Props. II. and III.

237. *To find a meaning for* $a^{\frac{p}{q}}$, p *and* q *being positive integers.*

Since $a^m \times a^n = a^{m+n}$ is to be true for *all* values of m and n, by replacing each of the indices m and n by $\frac{p}{q}$, we have

$$a^{\frac{p}{q}} \times a^{\frac{p}{q}} = a^{\frac{p}{q}+\frac{p}{q}} = a^{\frac{2p}{q}}.$$

Similarly, $a^{\frac{p}{q}} \times a^{\frac{p}{q}} \times a^{\frac{p}{q}} = a^{\frac{2p}{q}} \times a^{\frac{p}{q}} = a^{\frac{2p}{q}+\frac{p}{q}} = a^{\frac{3p}{q}}$.

Proceeding in this way for 4, 5, ... q factors, we have

$$a^{\frac{p}{q}} \times a^{\frac{p}{q}} \times a^{\frac{p}{q}} \text{ ...... to } q \text{ factors} = a^{\frac{qp}{q}};$$

that is,
$$(a^{\frac{p}{q}})^q = a^p.$$

Therefore, by taking the q^{th} root,

$$a^{\frac{p}{q}} = \sqrt[q]{a^p}.$$

or, in words, $a^{\frac{p}{q}}$ is equal to "the q^{th} root of a^p."

Examples. (1) $x^{\frac{5}{7}} = \sqrt[7]{x^5}$,

(2) $a^{\frac{1}{3}} = \sqrt[3]{a}$.

(3) $4^{\frac{3}{2}} = \sqrt{4^3} = \sqrt{64} = 8$.

(4) $a^{\frac{2}{3}} \times a^{\frac{5}{6}} = a^{\frac{2}{3}+\frac{5}{6}} = a^{\frac{3}{2}}$.

(5) $k^{\frac{a}{2}} \times k^{\frac{2}{3}} = k^{\frac{a}{2}+\frac{2}{3}} = k^{\frac{3a+4}{6}}$.

(6) $3a^{\frac{2}{3}}b^{\frac{1}{2}} \times 4a^{\frac{1}{6}}b^{\frac{5}{6}} = 12a^{\frac{2}{3}+\frac{1}{6}}b^{\frac{1}{2}+\frac{5}{6}} = 12a^{\frac{5}{6}}b^{\frac{4}{3}}$.

238. *To find a meaning for* a^0.

Since $a^m \times a^n = a^{m+n}$ is to be true for *all* values of m and n, by replacing the index m by 0, we have

$$a^0 \times a^n = a^{0+n}$$
$$= a^n;$$
$$\therefore \quad a^0 = \frac{a^n}{a^n}$$
$$= 1.$$

Hence *any quantity* with zero index is equivalent to 1.

Example. $x^{b-c} \times x^{c-b} = x^{b-c+c-b} = x^0 = 1$.

239. *To find a meaning for* a^{-n}.

Since $a^m \times a^n = a^{m+n}$ is to be true for *all* values of m and n, by replacing the index m by $-n$, we have

$$a^{-n} \times a^n = a^{-n+n} = a^0.$$

But
$$a^0 = 1;$$

hence $$a^{-n}=\frac{1}{a^n},$$

and $$a^n=\frac{1}{a^{-n}}.$$

From this it follows that any *factor* may be transferred from the numerator to the denominator of an expression, or vice-versâ, by merely changing the sign of the index.

Examples. (1) $x^{-3}=\dfrac{1}{x^3}.$

(2) $\dfrac{1}{y^{-\frac{1}{2}}}=y^{\frac{1}{2}}=\sqrt{y}.$

(3) $27^{-\frac{2}{3}}=\dfrac{1}{27^{\frac{2}{3}}}=\dfrac{1}{\sqrt[3]{(27)^2}}=\dfrac{1}{\sqrt[3]{3^6}}=\dfrac{1}{3^2}=\dfrac{1}{9}.$

240. *To prove that* $a^m\div a^n=a^{m-n}$ *for all values of* m *and* n.

$$\begin{aligned} a^m\div a^n&=a^m\times\frac{1}{a^n}\\ &=a^m\times a^{-n}\\ &=a^{m-n}, \text{ by the fundamental law.}\end{aligned}$$

Examples. (1) $a^3\div a^5=a^{3-5}=a^{-2}=\dfrac{1}{a^2}.$

(2) $c\div c^{-\frac{8}{5}}=c^{1+\frac{8}{5}}=c^{\frac{13}{5}}.$

(3) $x^{a-b}\div x^{a-c}=x^{a-b-(a-c)}=x^{c-b}.$

241. The method of finding a meaning for a symbol, as explained in the preceding articles, deserves careful attention. The usual algebraical process is to make choice of symbols, give them meanings, and then prove the rules for their combination. Here the process is reversed; the symbols are given, and the law to which they are to conform, and from this the meanings of the symbols are determined.

242. The following examples will illustrate the different principles we have established.

Examples. (1) $\dfrac{3a^{-2}}{5x^{-1}y}=\dfrac{3x}{5a^2y}.$

(2) $\dfrac{2a^{\frac{1}{2}}\times a^{\frac{2}{3}}\times 6a^{-\frac{7}{3}}}{9a^{-\frac{5}{3}}\times a^{\frac{3}{2}}}=\dfrac{4}{3}a^{\frac{1}{2}+\frac{2}{3}-\frac{7}{3}+\frac{5}{3}-\frac{3}{2}}=\dfrac{4}{3}a^{-1}=\dfrac{4}{3a}.$

(3) $\dfrac{\sqrt{x^3} \times \sqrt[3]{y^2}}{\sqrt[6]{y^{-2}} \times \sqrt[4]{x^6}} = \dfrac{x^{\frac{3}{2}} \times y^{\frac{2}{3}}}{y^{-\frac{1}{3}} \times x^{\frac{3}{2}}} = x^{\frac{3}{2}-\frac{3}{2}} y^{\frac{2}{3}+\frac{1}{3}} = x^0 y = y.$

(4) $2\sqrt{a} + \dfrac{3}{a^{-\frac{1}{2}}} + a^{\frac{5}{2}} = 2a^{\frac{1}{2}} + 3a^{\frac{1}{2}} + a^{\frac{5}{2}}$

$$= 5a^{\frac{1}{2}} + a^{\frac{5}{2}} = a^{\frac{1}{2}}(5 + a^2).$$

EXAMPLES XXX. a.

Express with positive indices :

1. $2x^{-\frac{1}{4}}$.
2. $3a^{-\frac{2}{3}}$.
3. $4x^{-2}a^3$.
4. $3 \div a^{-2}$.
5. $\dfrac{1}{4a^{-2}}$.
6. $\dfrac{1}{5x^{-\frac{1}{2}}}$.
7. $\dfrac{3a^{-3}x^2}{5y^2c^{-4}}$.
8. $\dfrac{x^a y^{-b}}{b^{-a}}$.
9. $2x^{\frac{1}{2}} \times 3x^{-1}$.
10. $1 \div 2a^{-\frac{1}{2}}$.
11. $xy^2 \times x^{-1}$.
12. $a^{-2}x^{-1} \div 3x$.
13. $\dfrac{1}{\sqrt{x^3}}$.
14. $\dfrac{1}{4\sqrt[5]{x^{-3}}}$.
15. $\dfrac{2}{\sqrt{y^{-3}}}$.
16. $\dfrac{\sqrt[4]{x^3}}{\sqrt{x^{-1}}}$.
17. $a^{-2}x^{-\frac{1}{2}} \div a^{-3}$.
18. $\sqrt[3]{a^{-1}} \div \sqrt[3]{a}$.
19. $\sqrt[5]{a^{-3}} \div \sqrt[5]{a^7}$.

Express with radical signs and positive indices :

20. $x^{\frac{3}{5}}$.
21. $a^{-\frac{1}{2}}$.
22. $5x^{-\frac{1}{2}}$.
23. $2a^{-\frac{1}{x}}$.
24. $\dfrac{1}{2a^{\frac{1}{3}}}$.
25. $\dfrac{2}{b^{-\frac{3}{4}}}$.
26. $\dfrac{c^{-\frac{1}{3}}}{2}$.
27. $\dfrac{1}{x^{-\frac{1}{x}}}$.
28. $a^{-\frac{1}{3}} \times 2a^{-\frac{1}{2}}$.
29. $x^{-\frac{2}{3}} \div 2a^{-\frac{1}{2}}$.
30. $7a^{-\frac{1}{2}} \times 3a^{-1}$.
31. $\dfrac{2a^{-2}}{a^{-\frac{3}{2}}}$.
32. $\dfrac{a^{-\frac{1}{2}}}{3a}$.
33. $\dfrac{4x^{-1}}{x^{-\frac{1}{3}}}$.
34. $\dfrac{\sqrt[3]{x^{-a}}}{\sqrt[3]{x^2}}$.
35. $\sqrt[3]{a^2} \times \sqrt[2]{a^3}$.
36. $\sqrt[5]{a^{-x}} \div \sqrt[5]{a^{-2x}}$.
37. $\sqrt[2a]{x} \times \sqrt[a]{x^2}$.
38. $\sqrt[a]{x} \div \sqrt[2a]{x^3}$.
39. $\sqrt[3x]{a^3} \div \sqrt[x]{a^2}$.
40. $\sqrt[4]{a^n} \times \sqrt[3]{a^n} \div \sqrt[12]{a^{5n}}$.

Find the value of

41. $16^{\frac{3}{4}}$. **42.** $4^{-\frac{5}{2}}$. **43.** $125^{\frac{2}{3}}$. **44.** $8^{-\frac{2}{3}}$. **45.** $36^{-\frac{3}{2}}$.

46. $\dfrac{1}{25^{-2}}$. **47.** $243^{\frac{2}{5}}$. **48.** $\left(\dfrac{8}{27}\right)^{-\frac{1}{3}}$. **49.** $\left(\dfrac{81}{16}\right)^{\frac{3}{4}}$. **50.** $\left(\dfrac{32}{243}\right)^{-\frac{7}{5}}$.

243. *To prove that* $(\mathrm{a}^{\mathrm{m}})^{\mathrm{n}}=\mathrm{a}^{\mathrm{mn}}$ *is universally true for all values of* m *and* n.

CASE I. Let n be a *positive integer.*

Now, *whatever be the value of* m

$$\begin{aligned}(a^m)^n &= a^m \cdot a^m \cdot a^m \text{ ...... to } n \text{ factors}\\ &= a^{m+m+m+\text{ ...... to } n \text{ terms}}\\ &= a^{mn}.\end{aligned}$$

CASE II. Let m be unrestricted as before, and let n be a *positive fraction.* Replacing n by $\dfrac{p}{q}$, where p and q are *positive integers,* we have $(a^m)^n=(a^m)^{\frac{p}{q}}$.

Now the q^{th} power of $(a^m)^{\frac{p}{q}}=\{(a^m)^{\frac{p}{q}}\}^q$

$$\begin{aligned}&=(a^m)^{\frac{p}{q}\cdot q}, &&\text{[Case I.]}\\ &=(a^m)^p\\ &=a^{mp}. &&\text{[Case I.]}\end{aligned}$$

Hence by taking the q^{th} root of these equals,

$$\begin{aligned}(a^m)^{\frac{p}{q}} &= \sqrt[q]{a^{mp}}\\ &= a^{\frac{mp}{q}}. &&\text{[Art. 237.]}\end{aligned}$$

CASE III. Let m be unrestricted as before, and let n be *any negative quantity.* Replacing n by $-r$, where r is *positive,* we have

$$\begin{aligned}(a^m)^n=(a^m)^{-r} &= \frac{1}{(a^m)^r}, &&\text{[Art. 239.]}\\ &= \frac{1}{a^{mr}}, &&\text{[Case II.]}\\ &= a^{-mr}=a^{mn}.\end{aligned}$$

Hence Prop. III., Art. 235, $(a^m)^n=a^{mn}$ has been shewn to be universally true.

Examples. (1) $(b^{\frac{2}{3}})^{\frac{6}{7}} = b^{\frac{2}{3}\times\frac{6}{7}} = b^{\frac{4}{7}}$.

(2) $\{(x^{-2})^3\}^{-4} = (x^{-6})^{-4} = x^{24}$.

(3) $\left(x^{\frac{1}{a-c}}\right)^{a^2-c^2} = x^{\frac{1}{a-c}\times(a^2-c^2)} = x^{a+c}$.

244. *To prove that* $(\mathrm{ab})^{\mathrm{n}} = \mathrm{a}^{\mathrm{n}}\mathrm{b}^{\mathrm{n}}$, *whatever be the value of* n; a *and* b *being any quantities whatever.*

CASE I. Let n be a *positive integer.*

Now $(ab)^n = ab \,.\, ab \,.\, ab \ldots\ldots$ to n factors

$$= (a \,.\, a \,.\, a \ldots \text{ to } n \text{ factors})(b \,.\, b \,.\, b \ldots \text{ to } n \text{ factors})$$

$$= a^n b^n.$$

CASE II. Let n be a *positive fraction.* Replacing n by $\frac{p}{q}$, where p and q are *positive integers,* we have $(ab)^n = (ab)^{\frac{p}{q}}$.

Now the q^{th} power of $(ab)^{\frac{p}{q}} = \{(ab)^{\frac{p}{q}}\}^q$

$$= (ab)^p, \qquad \text{[Art. 243.]}$$

$$= a^p b^p$$

$$= (a^{\frac{p}{q}} b^{\frac{p}{q}})^q. \qquad \text{[Case I.]}$$

Taking the q^{th} root, $(ab)^{\frac{p}{q}} = a^{\frac{p}{q}} b^{\frac{p}{q}}$.

CASE III. Let n have *any negative value.* Replacing n by $-r$, where r is positive,

$$(ab)^n = (ab)^{-r} = \frac{1}{(ab)^r}$$

$$= \frac{1}{a^r b^r} = a^{-r} b^{-r}$$

$$= a^n b^n.$$

Hence the proposition is proved universally.

The result we have just proved may be expressed in a verbal form by saying that the index of a product may be *distributed* over its *factors.*

Note. An index is not distributive over the *terms* of an expression. Thus $(a^{\frac{1}{2}} + b^{\frac{1}{2}})^2$ is not equal to $a+b$. Again $(a^2+b^2)^{\frac{1}{2}}$ is equal to $\sqrt{a^2+b^2}$, and cannot be further simplified.

Examples. (1) $(yz)^{a-c}(zx)^c(xy)^{-c} = y^{a-c}z^{a-c}z^c x^c x^{-c}y^{-c}$

$$= y^{a-2c}z^a.$$

(2) $\{(a-b)^k\}^{-l} \times \{(a+b)^{-k}\}^l = (a-b)^{-kl} \times (a+b)^{-kl}$

$$= \{(a-b)(a+b)\}^{-kl}$$

$$= (a^2-b^2)^{-kl}.$$

245. It should be observed that in the proof of Art. 244 the quantities a and b are *wholly unrestricted*, and may themselves involve indices.

Examples. (1) $\left(x^{\frac{1}{2}}y^{-\frac{1}{2}}\right)^{\frac{4}{3}} \div (x^2y^{-1})^{-\frac{1}{3}} = x^{\frac{2}{3}}y^{-\frac{2}{3}} \div x^{-\frac{2}{3}}y^{\frac{1}{3}}$

$$= x^{\frac{4}{3}}y^{-1}.$$

(2) $\left(\dfrac{a^{\frac{2}{3}}\sqrt{b^{-1}}}{b\sqrt[3]{a^{-2}}} \div \sqrt{\dfrac{a\sqrt{b^{-4}}}{b\sqrt{a^{-2}}}}\right)^6 = \left(\dfrac{a^{\frac{2}{3}}b^{-\frac{1}{2}}}{ba^{-\frac{2}{3}}} \div \sqrt{\dfrac{ab^{-2}}{ba^{-1}}}\right)^6$

$$= \left(a^{\frac{4}{3}}b^{-\frac{3}{2}} \div \sqrt{a^2b^{-3}}\right)^6$$

$$= \left(a^{\frac{4}{3}}b^{-\frac{3}{2}} \div ab^{-\frac{3}{2}}\right)^6$$

$$= \left(a^{\frac{1}{3}}\right)^6 = a^2.$$

EXAMPLES XXX. b.

Simplify and express with positive indices :

1. $(\sqrt{a^2b^3})^6$.
2. $(\sqrt[9]{x^{-4}y^3})^{-3}$.
3. $(x^ay^{-b})^3 \times (x^3y^2)^{-a}$
4. $\left(\dfrac{16x^2}{y^{-2}}\right)^{-\frac{1}{4}}$.
5. $\left(\dfrac{27x^3}{8a^{-3}}\right)^{-\frac{2}{3}}$.
6. $\left(\dfrac{a^{-\frac{1}{2}}}{4c^2}\right)^{-2}$.
7. $\left\{\sqrt[4]{\left(x^{-\frac{2}{3}}y^{\frac{1}{2}}\right)^3}\right\}^{-\frac{2}{3}}$.
8. $\sqrt[4]{x\sqrt[3]{x^{-1}}}$.
9. $(4a^{-2} \div 9x^2)^{-\frac{1}{2}}$.
10. $(x \div \sqrt[n]{x})^n$.
11. $\left(x \times \sqrt[n]{x^{-\frac{1}{n}}}\right)^{\frac{n^2}{1-n}}$.
12. $(\sqrt[b]{x^b} \div \sqrt[a]{x})^{\frac{1}{1-a}}$
13. $\sqrt{a^{-2}b} \times \sqrt[3]{ab^{-3}}$.
14. $\sqrt[3]{ab^{-1}c^{-2}} \times (a^{-1}b^{-2}c^{-4})^{-\frac{1}{6}}$.
15. $\sqrt[6]{a^{4b}x^6} \times (a^{\frac{2}{3}}x^{-1})^{-b}$.
16. $\sqrt[3]{x^{-1}\sqrt{y^3}} \div \sqrt{y\sqrt[3]{x}}$.

Simplify and express with positive indices:

17. $(a^{-\frac{1}{2}}\sqrt[3]{x})^{-3} \times \sqrt{x^{-2}\sqrt{a^{-6}}}$.

18. $\sqrt[n]{a^{n+k}b^{2n-k}} \div (a^{\frac{1}{n}}b^{-\frac{1}{n}})^k$.

19. $\sqrt[3]{(a+b)^5} \times (a+b)^{-\frac{2}{3}}$.

20. $\{(x-y)^{-3}\}^n \div \{(x+y)^n\}^3$.

21. $\left(\dfrac{a^{-2}b}{a^3b^{-4}}\right)^{-3} \div \left(\dfrac{ab^{-1}}{a^{-3}b^2}\right)^5$.

22. $\left\{\dfrac{\sqrt[3]{a}}{\sqrt[4]{b^{-1}}} \cdot \left(\dfrac{b^{\frac{1}{4}}}{a^{\frac{1}{3}}}\right)^2 \div \dfrac{a^{-\frac{1}{3}}}{b^{-\frac{1}{2}}}\right\}^6$.

23. $\left(a^{-\frac{1}{2}}x^{\frac{1}{3}}\sqrt{ax^{-\frac{1}{3}}\sqrt[4]{x^{\frac{4}{3}}}}\right)^{\frac{1}{3}}$.

24. $\sqrt[4]{(a+b)^6} \times (a^2-b^2)^{-\frac{1}{2}}$.

25. $\left(\dfrac{a^{-3}}{b^{-\frac{2}{3}}c}\right)^{-\frac{3}{2}} \div \left(\dfrac{\sqrt{a^{-\frac{1}{2}} \cdot \sqrt[6]{b^3}}}{a^2c^{-1}}\right)^{-2}$.

26. $\left(\dfrac{a^{-\frac{2}{3}}x^{\frac{1}{2}}}{x^{-1}a}\right)^2 \div \sqrt[3]{\dfrac{a^{-1}}{x^{-3}}}$.

27. $\left(\sqrt[5]{\dfrac{a^{\frac{1}{2}}x^{-2}}{x^{\frac{1}{2}}a^{-2}}} \times \sqrt[3]{\dfrac{a\sqrt{x}}{x^{-1}\sqrt{a}}}\right)^{-4}$.

28. $\dfrac{\sqrt[3]{(a^3b^3+a^6)}}{\sqrt[3]{(b^6-a^3b^3)^{-1}}}$.

29. $(a^{n^2-1})^{\frac{n}{n+1}} + \dfrac{\sqrt[n]{a^{2n}}}{a}$.

30. $\left(x^{\frac{n}{n+1}}\right)^{n^2-1} + \dfrac{\sqrt{x^{2n}}}{x}$.

31. $\left\{\dfrac{a^{p-q}}{\sqrt[q]{a^{q^2-pq}}} \times a^{2(p-q)}\right\}^n$.

32. $(x^{\frac{a}{b}}y^{-1})^b \div \left(\dfrac{x^{a^2-b^2}}{y^{ab+b^2}}\right)^{\frac{1}{a+b}}$.

33. $\left(\dfrac{x^{-2}y^3}{x^3y^{-2}}\right)^{-\frac{1}{5}} \times \left(\dfrac{y^3x^{-3}}{x^3y^{-3}}\right)^{-1}$.

34. $\left(\dfrac{y^{-3}}{x^{\frac{2}{7}}z^{-1}}\right)^{-\frac{3}{2}} \times \left(\dfrac{y^{\frac{14}{3}}x^{-1}}{z^{-\frac{21}{4}}}\right)^{\frac{2}{7}}$.

35. $\dfrac{2^n \times (2^{n-1})^n}{2^{n+1} \times 2^{n-1}} \times \dfrac{1}{4^{-n}}$.

36. $\dfrac{2^{n+1}}{(2^n)^{n-1}} \div \dfrac{4^{n+1}}{(2^{n-1})^{n+1}}$.

37. $\dfrac{3 \cdot 2^n - 4 \cdot 2^{n-2}}{2^n - 2^{n-1}}$.

38. $\dfrac{3^{n+4} - 6 \cdot 3^{n+1}}{3^{n+2} \times 7}$.

246. Since the index-laws are universally true, all the ordinary operations of multiplication, division, involution and evolution are applicable to expressions which contain fractional and negative indices.

247. In Art. 121, we pointed out that the descending powers of x are

$$\ldots\ldots\ x^3,\ x^2,\ x,\ 1,\ \frac{1}{x},\ \frac{1}{x^2},\ \frac{1}{x^3},\ \ldots\ldots$$

A reason for this may be seen if we write these terms in the form

$$\ldots\ldots\ x^3,\ x^2,\ x^1,\ x^0,\ x^{-1},\ x^{-2},\ x^{-3},\ \ldots\ldots$$

Example 1. Multiply $3x^{-\frac{1}{3}}+x+2x^{\frac{2}{3}}$ by $x^{\frac{1}{3}}-2$.

Arrange in descending powers of x.

$$
\begin{array}{llll}
x & +2x^{\frac{2}{3}} & +3x^{-\frac{1}{3}} & \\
x^{\frac{1}{3}}-2 & & & \\
\hline
x^{\frac{4}{3}}+2x & +3 & & \\
\quad -2x & -4x^{\frac{2}{3}} & -6x^{-\frac{1}{3}} & \\
\hline
x^{\frac{4}{3}} & -4x^{\frac{2}{3}} & +3 & -6x^{-\frac{1}{3}}.
\end{array}
$$

Example 2. Divide $16a^{-3}-6a^{-2}+5a^{-1}+6$ by $1+2a^{-1}$.

$$
\begin{array}{l}
2a^{-1}+1\,)\,16a^{-3}-\ 6a^{-2}+\ 5a^{-1}+6\,(\,8a^{-2}-7a^{-1}+6 \\
\qquad\qquad \underline{16a^{-3}+\ 8a^{-2}} \\
\qquad\qquad\qquad -14a^{-2}+\ 5a^{-1} \\
\qquad\qquad\qquad \underline{-14a^{-2}-\ 7a^{-1}} \\
\qquad\qquad\qquad\qquad\qquad 12a^{-1}+6 \\
\qquad\qquad\qquad\qquad\qquad \underline{12a^{-1}+6}
\end{array}
$$

Example 3. Find the square root of

$$\frac{4x^2}{y}+\frac{\sqrt{x^3}}{y^{-\frac{1}{2}}}-2x+\frac{y}{4}+x^3-4\sqrt{(x^5y^{-1})}.$$

Getting rid of the radical signs, and arranging in descending powers of x, we have

$$
\begin{array}{l|l}
 & x^3-4x^{\frac{5}{2}}y^{-\frac{1}{2}}+4x^2y^{-1}+x^{\frac{3}{2}}y^{\frac{1}{2}}-2x+\frac{y}{4}\left(x^{\frac{3}{2}}-2xy^{-\frac{1}{2}}+\frac{y^{\frac{1}{2}}}{2}\right. \\
 & \underline{x^3} \\
2x^{\frac{3}{2}}-2xy^{-\frac{1}{2}} & -4x^{\frac{5}{2}}y^{-\frac{1}{2}}+4x^2y^{-1} \\
 & \underline{-4x^{\frac{5}{2}}y^{-\frac{1}{2}}+4x^2y^{-1}} \\
2x^{\frac{3}{2}}-4xy^{-\frac{1}{2}}+\frac{y^{\frac{1}{2}}}{2} & \qquad\qquad\qquad\quad x^{\frac{3}{2}}y^{\frac{1}{2}}-2x+\frac{y}{4} \\
 & \qquad\qquad\qquad\quad \underline{x^{\frac{3}{2}}y^{\frac{1}{2}}-2x+\frac{y}{4}}
\end{array}
$$

Note. In this example it should be observed that the introduction of negative indices enables us to avoid the use of algebraical fractions.

EXAMPLES XXX. c.

1. Multiply $3x^{\frac{1}{3}} - 5 + 8x^{-\frac{1}{3}}$ by $4x^{\frac{1}{3}} + 3x^{-\frac{1}{3}}$.

2. Multiply $3a^{\frac{3}{5}} - 4a^{\frac{1}{5}} - a^{-\frac{1}{5}}$ by $3a^{\frac{1}{5}} + a^{-\frac{1}{5}} - 6a^{-\frac{3}{5}}$.

3. Find the product of $c^x + 2c^{-x} - 7$ and $5 - 3c^{-x} + 2c^x$.

4. Find the product of $5 + 2x^{2a} + 3x^{-2a}$ and $4x^a - 3x^{-a}$.

5. Divide $21x + x^{\frac{2}{3}} + x^{\frac{1}{3}} + 1$ by $3x^{\frac{1}{3}} + 1$.

6. Divide $15a - 3a^{\frac{1}{3}} - 2a^{-\frac{1}{3}} + 8a^{-1}$ by $5a^{\frac{2}{3}} + 4$.

7. Divide $16a^{-3} + 6a^{-2} + 5a^{-1} - 6$ by $2a^{-1} - 1$.

8. Divide $5b^{\frac{2}{3}} - 6b^{\frac{1}{3}} - 4b^{-\frac{2}{3}} - 4b^{-\frac{1}{3}} - 5$ by $b^{\frac{1}{6}} - 2b^{-\frac{1}{6}}$.

9. Divide $21a^{3x} + 20 - 27a^x - 26a^{2x}$ by $3a^x - 5$.

10. Divide $8c^{-n} - 8c^n + 5c^{3n} - 3c^{-3n}$ by $5c^n - 3c^{-n}$.

Find the square root of

11. $9x - 12x^{\frac{1}{2}} + 10 - 4x^{-\frac{1}{2}} + x^{-1}$.

12. $25a^{\frac{4}{3}} + 16 - 30a - 24a^{\frac{1}{3}} + 49a^{\frac{2}{3}}$.

13. $4x^n + 9x^{-n} + 28 - 24x^{-\frac{n}{2}} - 16x^{\frac{n}{2}}$.

14. $12a^x + 4 - 6a^{3x} + a^{4x} + 5a^{2x}$.

15. Multiply $a^{\frac{3}{2}} - 8a^{-\frac{3}{2}} + 4a^{-\frac{1}{2}} - 2a^{\frac{1}{2}}$ by $4a^{-\frac{3}{2}} + a^{\frac{1}{2}} + 4a^{-\frac{1}{2}}$.

16. Multiply $1 - 2\sqrt[3]{x} - 2x^{\frac{1}{2}}$ by $1 - \sqrt[6]{x}$.

17. Multiply $2\sqrt[3]{a^5} - a^{\frac{4}{3}} - \frac{3}{a}$, by $2a - 3\sqrt[3]{\frac{1}{a}} - a^{-\frac{5}{3}}$.

18. Divide $\sqrt[3]{x^2} + 2x^{\frac{1}{3}} - 16x^{-\frac{2}{3}} - \frac{32}{x}$ by $x^{\frac{1}{6}} + 4x^{-\frac{1}{6}} + \frac{4}{\sqrt{x}}$.

19. Divide $1 - \sqrt{a} - \frac{2}{a^{-1}} + 2a^2$ by $1 - a^{\frac{1}{2}}$.

20. Divide $4\sqrt[3]{x^2} - 8x^{\frac{1}{3}} - 5 + \frac{10}{\sqrt[3]{x}} + 3x^{-\frac{2}{3}}$ by $2x^{\frac{5}{12}} - \sqrt[12]{x} - \frac{3}{\sqrt[4]{x}}$.

Find the square root of

21. $9x^{-4} - 18x^{-3}\sqrt{y} + \frac{15y}{x^2} - 6\sqrt{\left(\frac{y^3}{x^2}\right)} + y^2.$

22. $4\sqrt{x^3} - 12\sqrt[4]{(x^3y)} + 25\sqrt{y} - 24\sqrt[4]{\left(\frac{y^3}{x^3}\right)} + 16x^{-\frac{3}{2}}y.$

23. $81\left(\frac{\sqrt[3]{x^4}}{y^2} + 1\right) + 36\frac{x^{\frac{1}{3}}}{\sqrt{y}}(x^{\frac{2}{3}}y^{-1} - 1) - 158\frac{\sqrt[3]{x^2}}{y}.$

24. $\frac{x^{-2}}{16} + 1 + \frac{9}{\sqrt[3]{y^{-2}}} + \frac{1 - 3\sqrt[3]{y}}{2x} - 6\sqrt[3]{y}.$

248. The following examples will illustrate the formulæ of earlier chapters when applied to expressions involving fractional and negative indices.

Example 1.
$$(a^{\frac{h}{k}} - b^{\frac{p}{q}})(a^{-\frac{h}{k}} + b^{-\frac{p}{q}}) = a^{\frac{h}{k}-\frac{h}{k}} - a^{-\frac{h}{k}}b^{\frac{p}{q}} + a^{\frac{h}{k}}b^{-\frac{p}{q}} - b^{\frac{p}{q}-\frac{p}{q}}$$
$$= 1 - a^{-\frac{h}{k}}b^{\frac{p}{q}} + a^{\frac{h}{k}}b^{-\frac{p}{q}} - 1$$
$$= a^{\frac{h}{k}}b^{-\frac{p}{q}} - a^{-\frac{h}{k}}b^{\frac{p}{q}}.$$

Example 2. Multiply $2x^{2p} - x^p + 3$ by $2x^{2p} + x^p - 3$.

The product $= \{2x^{2p} - (x^p - 3)\}\{2x^{2p} + (x^p - 3)\}$
$$= (2x^{2p})^2 - (x^p - 3)^2$$
$$= 4x^{4p} - x^{2p} + 6x^p - 9.$$

Example 3. The square of $3x^{\frac{1}{2}} - 2 - x^{-\frac{1}{2}}$
$$= 9x + 4 + x^{-1} - 2 \cdot 3x^{\frac{1}{2}} \cdot 2 - 2 \cdot 3x^{\frac{1}{2}} \cdot x^{-\frac{1}{2}} + 2 \cdot 2 \cdot x^{-\frac{1}{2}}$$
$$= 9x + 4 + x^{-1} - 12x^{\frac{1}{2}} - 6 + 4x^{-\frac{1}{2}}$$
$$= 9x - 12x^{\frac{1}{2}} - 2 + 4x^{-\frac{1}{2}} + x^{-1},$$
by collecting like terms and rearranging.

Example 4. Divide $a^{\frac{3n}{2}} + a^{-\frac{3n}{2}}$ by $a^{\frac{n}{2}} + a^{-\frac{n}{2}}$.

The quotient $= (a^{\frac{3n}{2}} + a^{-\frac{3n}{2}}) \div (a^{\frac{n}{2}} + a^{-\frac{n}{2}})$
$$= \{(a^{\frac{n}{2}})^3 + (a^{-\frac{n}{2}})^3\} \div (a^{\frac{n}{2}} + a^{-\frac{n}{2}})$$
$$= (a^{\frac{n}{2}})^2 - a^{\frac{n}{2}} \cdot a^{-\frac{n}{2}} + (a^{-\frac{n}{2}})^2$$
$$= a^n - 1 + a^{-n}.$$

EXAMPLES XXX. d.

Write down the value of

1. $(x^{\frac{1}{2}}-7)(x^{\frac{1}{2}}+3)$.

2. $(4x-5x^{-1})(4x+3x^{-1})$.

3. $(7x-9y^{-1})(7x+9y^{-1})$.

4. $(x^m-y^n)(x^{-m}+y^{-n})$.

5. $(a^x-2a^{-x})^2$.

6. $(a^x+a^{\frac{1}{x}})^2$.

7. $\left(x^{\frac{a}{2}}-\frac{1}{2}x^{-a}\right)^2$.

8. $(5x^ay^b-3x^{-a}y^{-b})(4x^ay^b+5x^{-a}y^{-b})$.

9. $\left(\frac{1}{3}a^{\frac{1}{3}}-a^{-\frac{1}{3}}\right)^2$.

10. $(3x^ay^{-b}+5x^{-a}y^b)(3x^ay^b-5x^{-a}y^{-b})$

11. $\left(a^x-\frac{1}{2}-a^{-x}\right)^2$.

12. $(x^{\frac{1}{a}}-x^{-\frac{1}{a}}+x)^2$.

13. $\{(a+b)^{\frac{1}{2}}+(a-b)^{\frac{1}{2}}\}^2$.

14. $\{(a+b)^{\frac{1}{2}}-(a-b)^{-\frac{1}{2}}\}^2$.

Write down the quotient of

15. $x-9a$ by $x^{\frac{1}{2}}+3a^{\frac{1}{2}}$.

16. $x^{\frac{3}{2}}-27$ by $x^{\frac{1}{2}}-3$.

17. $a^{2x}-16$ by a^x-4.

18. $x^{3a}+8$ by x^a+2.

19. $c^{2x}-c^{-x}$ by $c^x-c^{-\frac{x}{2}}$.

20. $1-8a^{-3}$ by $1-2a^{-1}$.

21. $a^{4x}-x^6$ by $a^{2x}+x^3$.

22. $x^{-4}-1$ by $x^{-1}+1$.

23. $x^{\frac{5}{3}}-1$ by $x^{\frac{1}{3}}-1$.

24. $x^{5n}+32$ by x^n+2.

Find the value of

25. $(x+x^{\frac{1}{2}}-4)(x+x^{\frac{1}{2}}+4)$.

26. $(2x^{\frac{1}{3}}+4+3x^{-\frac{1}{3}})(2x^{\frac{1}{3}}+4-3x^{-\frac{1}{3}})$.

27. $(2-x^{\frac{1}{3}}+x)(2+x^{\frac{1}{3}}+x)$.

28. $(a^x+7+3a^{-x})(a^x-7-3a^{-x})$.

29. $\dfrac{a^{\frac{4}{3}}-8a^{\frac{1}{3}}b}{a^{\frac{2}{3}}+2\sqrt[3]{ab}+4b^{\frac{2}{3}}}$.

30. $\dfrac{x-7x^{\frac{1}{2}}}{x-5\sqrt{x}-14}\div\left(1+\dfrac{2}{\sqrt{x}}\right)^{-1}$.

31. $\dfrac{x^{\frac{2}{3}}-4\sqrt[3]{x^{-2}}}{\sqrt[3]{x^2}+4+4x^{-\frac{2}{3}}}$.

32. $\dfrac{a^{\frac{3}{2}}+ab}{ab-b^3}-\dfrac{\sqrt{a}}{\sqrt{a}-b}$

CHAPTER XXXI.

ELEMENTARY SURDS.

249. DEFINITION. If the root of a quantity cannot be exactly obtained the root is called a **surd.**

Thus $\sqrt{2}$, $\sqrt[3]{5}$, $\sqrt[5]{a^3}$, $\sqrt{a^2+b^2}$ are surds.

By reference to the preceding chapter it will be seen that these are only cases of fractional indices; for the above quantities might be written

$$2^{\frac{1}{2}},\ 5^{\frac{1}{3}},\ a^{\frac{3}{5}},\ (a^2+b^2)^{\frac{1}{2}}.$$

Since surds may always be expressed as quantities with fractional indices they are subject to the same laws of combination as other algebraical symbols.

250. A quantity may be expressed in a surd form without really being a surd. Thus $\sqrt[3]{x^6}$ or $x^{\frac{6}{3}}$, though apparently a surd, can be expressed in the equivalent form x^2.

251. A surd is sometimes called an **irrational quantity**; and quantities which are not surds are, for the sake of distinction, termed **rational quantities.**

252. In the case of numerical surds such as $\sqrt{2}$, $\sqrt[3]{5}$, ..., although the *exact* value can never be found, it can be determined to any degree of accuracy by carrying the process of evolution far enough.

Thus $\qquad\qquad \sqrt{5} = 2{\cdot}236068............$;

that is $\sqrt{5}$ lies between 2·23606 and 2·23607; and therefore the error in using either of these quantities instead of $\sqrt{5}$ is less than ·00001. By taking the root to a greater number of decimal places we can approximate still nearer to the true value.

It thus appears that it will never be *absolutely necessary* to introduce surds into numerical work, which can always be carried on to a certain degree of accuracy; but we shall in the present chapter prove laws for combination of surd quantities which will enable us to work with symbols such as $\sqrt{2}$, $\sqrt[3]{5}$, $\sqrt[4]{a}$, ... with absolute accuracy so long as the symbols are kept in their surd form. Moreover it will be found that even where approximate numerical results are required, the work is considerably simpli-

fied and shortened by operating with surd symbols, and afterwards substituting numerical values, if necessary.

253. The *order* of a surd is indicated by the root symbol, or surd index. Thus $\sqrt[3]{x}$, $\sqrt[n]{a}$ are respectively surds of the third and n^{th} orders.

The surds of the most frequent occurrence are those of the second order; they are sometimes called **quadratic surds.** Thus $\sqrt{3}$, $\sqrt{a}$, $\sqrt{x+y}$ are quadratic surds.

254. It will frequently be found convenient to express a rational quantity in a surd form.

A rational quantity may be expressed in the form of a surd of *any required order* by raising it to the power whose root the surd expresses, and prefixing the radical sign. Thus

$$5=\sqrt{25}=\sqrt[3]{125}=\sqrt[4]{625}=\sqrt[n]{5^n};$$
$$a+x=\sqrt{(a+x)^2}=\sqrt[6]{(a+x)^6}=\sqrt[n]{(a+x)^n}.$$

255. A surd of any order may be transformed into a surd of a different order.

Examples. (1) $\sqrt[3]{2}=2^{\frac{1}{3}}=2^{\frac{4}{12}}=\sqrt[12]{2^4}$.

(2) $\sqrt[p]{a}=a^{\frac{1}{p}}=a^{\frac{q}{pq}}=\sqrt[pq]{a^q}$.

256. Surds of different orders may be transformed into surds of the same order. This order may be *any* common multiple of each of the given orders, but it is usually most convenient to choose the *least* common multiple.

Example. Express $\sqrt[4]{a^3}$, $\sqrt[3]{b^2}$, $\sqrt[6]{a^5}$ as surds of the same lowest order.

The least common multiple of 4, 3, 6 is 12; and expressing the given surds as surds of the twelfth order they become $\sqrt[12]{a^9}$, $\sqrt[12]{b^8}$, $\sqrt[12]{a^{10}}$.

257. Surds of different orders may be arranged according to magnitude by transforming them into surds of the same order.

Example. Arrange $\sqrt{3}$, $\sqrt[3]{6}$, $\sqrt[4]{10}$ according to magnitude.

The least common multiple of 2, 3, 4 is 12; and, expressing the given surds as surds of the twelfth order, we have

$$\sqrt{3}=\sqrt[12]{3^6}=\sqrt[12]{729},$$
$$\sqrt[3]{6}=\sqrt[12]{6^4}=\sqrt[12]{1296},$$
$$\sqrt[4]{10}=\sqrt[12]{10^3}=\sqrt[12]{1000}.$$

Hence arranged in ascending order of magnitude the surds are

$$\sqrt{3},\ \sqrt[4]{10},\ \sqrt[3]{6}.$$

EXAMPLES XXXI. a.

Express as surds of the twelfth order with positive indices:

1. $x^{\frac{1}{3}}$. 2. $a^{-1} \div a^{-\frac{1}{2}}$. 3. $\sqrt[4]{ax^3} \times \sqrt[3]{a^{-1}x^{-2}}$.

4. $\dfrac{1}{a^{-\frac{3}{4}}}$. 5. $\dfrac{1}{\sqrt[8]{a^{-14}}}$. 6. $\sqrt[6]{\dfrac{1}{a^{-2}}}$.

Express as surds of the n^{th} order with positive indices:

7. $\sqrt[3]{x^2}$. 8. x^a. 9. $a^{\frac{1}{2}}$. 10. $\sqrt{a^{-\frac{1}{n}}}$.

11. $\sqrt[3]{\dfrac{1}{x^n y^n}}$. 12. $\dfrac{1}{a^{-1}}$. 13. $\dfrac{x^{-\frac{1}{2}}}{y^2}$. 14. $\dfrac{a^{\frac{1}{2}}}{x^{-n}}$.

Express as surds of the same lowest order:

15. $\sqrt{a}$, $\sqrt[9]{a^5}$. 16. $\sqrt[5]{a^3}$, $\sqrt{a}$. 17. $\sqrt[8]{x^3}$, $\sqrt[9]{x^6}$, $\sqrt[20]{x^5}$.

18. $\sqrt[16]{x^4}$, $\sqrt[12]{x^{10}}$. 19. $\sqrt[21]{a^3b^4}$, $\sqrt[7]{ab}$. 20. $\sqrt{ax^2}$, $\sqrt[39]{a^9x^6}$.

21. $\sqrt{5}$, $\sqrt[3]{11}$, $\sqrt[6]{13}$. 22. $\sqrt[4]{8}$, $\sqrt{3}$, $\sqrt[8]{6}$. 23. $\sqrt[3]{2}$, $\sqrt[9]{8}$, $\sqrt[6]{4}$.

258. The root of any expression is equal to the product of the roots of the separate factors of the expression.

For
$$\sqrt[n]{ab} = (ab)^{\frac{1}{n}}$$
$$= a^{\frac{1}{n}} b^{\frac{1}{n}}, \qquad \text{[Art. 244}$$
$$= \sqrt[n]{a} \cdot \sqrt[n]{b}.$$

Similarly, $\sqrt[n]{abc} = \sqrt[n]{a} \cdot \sqrt[n]{b} \cdot \sqrt[n]{c}$;

and so for any number of factors.

Examples. (1) $\sqrt[4]{15} = \sqrt[4]{3} \cdot \sqrt[4]{5}$.

(2) $\sqrt[3]{a^6b} = \sqrt[3]{a^6} \cdot \sqrt[3]{b} = a^2\sqrt[3]{b}$.

(3) $\sqrt{50} = \sqrt{25} \cdot \sqrt{2} = 5\sqrt{2}$.

Hence it appears that a surd may sometimes be expressed as the product of a rational quantity and a surd; when so reduced the surd is said to be in its *simplest form.*

Thus the simplest form of $\sqrt{128}$ is $8\sqrt{2}$.

Conversely, the coefficient of a surd may be brought under the radical sign by first reducing it to the form of a surd, and then multiplying the surds together.

Examples. (1) $7\sqrt{5}=\sqrt{49}\,.\,\sqrt{5}=\sqrt{245}$.

(2) $a\sqrt[8]{b}=\sqrt[8]{a^8}\,.\,\sqrt[8]{b}=\sqrt[8]{a^8b}$.

When so reduced a surd is said to be an *entire surd.*

259. When surds have, or can be reduced to have, the same irrational factor, they are said to be *like*; otherwise, they are said to be *unlike.* Thus

$$5\sqrt{3},\ 2\sqrt{3},\ \frac{1}{5}\sqrt{3} \text{ are like surds.}$$

But $3\sqrt{2}$ and $2\sqrt{3}$ are unlike surds.

Again, $3\sqrt{20},\ 4\sqrt{5},\ \sqrt{\frac{1}{5}}$ are like surds;

for $3\sqrt{20}=3\sqrt{4}\,.\,\sqrt{5}=3\,.\,2\sqrt{5}=6\sqrt{5}$;

and $\sqrt{\frac{1}{5}}=\sqrt{\frac{5}{25}}=\frac{1}{5}\sqrt{5}$.

260. In finding the sum of a number of like surds we reduce them to their simplest form, and prefix to their common irrational part the sum of the coefficients.

Example 1. The sum of $3\sqrt{20},\ 4\sqrt{5},\ \frac{1}{\sqrt{5}}$

$$=6\sqrt{5}+4\sqrt{5}+\frac{1}{5}\sqrt{5}$$

$$=\frac{51}{5}\sqrt{5}.$$

Example 2. The sum of $x\sqrt[3]{8x^3a}+y\sqrt[3]{-y^3a}-z\sqrt[3]{z^3a}$

$$=x\,.\,2x\sqrt[3]{a}+y(-y)\sqrt[3]{a}-z\,.\,z\sqrt[3]{a}$$

$$=(2x^2-y^2-z^2)\sqrt[3]{a}.$$

261. Unlike surds cannot be collected.

Thus the sum of $5\sqrt{2}$, $-2\sqrt{3}$ and $\sqrt{6}$ is $5\sqrt{2}-2\sqrt{3}+\sqrt{6}$ and cannot be further simplified.

EXAMPLES XXXI. b.

Express in the simplest form:

1. $\sqrt{288}$.	**2.** $\sqrt{147}$.	**3.** $\sqrt[3]{256}$.	**4.** $\sqrt[3]{432}$.
5. $3\sqrt{150}$.	**6.** $2\sqrt{720}$.	**7.** $5\sqrt{245}$.	**8.** $\sqrt[3]{1029}$.
9. $\sqrt[4]{3125}$.	**10.** $\sqrt[3]{-2187}$.	**11.** $\sqrt{36a^3}$.	**12.** $\sqrt{27a^3b^5}$.

13. $\sqrt[3]{-108x^4y^3}$. 14. $\sqrt[n]{x^{3n}y^{2n+5}}$. 15. $\sqrt[p]{x^{a+p}y^{2p}}$.

16. $\sqrt{a^3+2a^2b+ab^2}$. 17. $\sqrt[3]{8x^4y-24x^3y^2+24x^2y^3-8xy^4}$.

Express as entire surds:

18. $11\sqrt{2}$. 19. $14\sqrt{5}$. 20. $6\sqrt[3]{4}$. 21. $5\sqrt[3]{6}$.

22. $\frac{4}{11}\sqrt{\frac{77}{8}}$. 23. $\frac{3ab}{2c}\sqrt{\frac{20c^2}{9a^2b}}$. 24. $\frac{3x}{y}\sqrt{\frac{a^2y^3}{x^2}}$.

25. $\frac{a}{x^2}\sqrt{\frac{3x^3}{a}}$. 26. $\frac{2a}{3x}\sqrt[3]{\frac{27x^4}{a^2}}$. 27. $\frac{2a}{b}\sqrt[4]{\frac{b^4}{8a^3}}$.

28. $a\sqrt[n]{\frac{b^2}{a^{n-2}}}$. 29. $\frac{a}{b}\sqrt[p]{\frac{b^{p+1}}{a^{p-1}}}$. 30. $\frac{y}{x^n}\sqrt{\frac{x^{2n+1}}{y^3}}$.

31. $(x+y)\sqrt{\frac{x-y}{x+y}}$. 32. $\frac{ax}{a-x}\sqrt{\frac{a^2-x^2}{a^2x^2}}$.

Find the value of

33. $3\sqrt{45}-\sqrt{20}+7\sqrt{5}$. 34. $4\sqrt{63}+5\sqrt{7}-8\sqrt{28}$.

35. $\sqrt{44}-5\sqrt{176}+2\sqrt{99}$. 36. $2\sqrt{363}-5\sqrt{243}+\sqrt{192}$.

37. $2\sqrt[3]{189}+3\sqrt[3]{875}-7\sqrt[3]{56}$. 38. $5\sqrt[3]{81}-7\sqrt[3]{192}+4\sqrt[3]{648}$.

39. $3\sqrt[4]{162}-7\sqrt[4]{32}+\sqrt[4]{1250}$. 40. $5\sqrt[3]{-54}-2\sqrt[3]{-16}+4\sqrt[3]{686}$.

41. $4\sqrt{128}+4\sqrt{75}-5\sqrt{162}$. 42. $5\sqrt{24}-2\sqrt{54}-\sqrt{6}$.

43. $\sqrt{252}-\sqrt{294}-48\sqrt{\frac{1}{6}}$. 44. $3\sqrt{147}-\frac{7}{3}\sqrt{\frac{1}{3}}-\sqrt{\frac{1}{27}}$.

262. *To multiply two surds of the same order: multiply separately the rational factors and the irrational factors.*

For
$$a\sqrt[n]{x}\times b\sqrt[n]{y}=ax^{\frac{1}{n}}\times by^{\frac{1}{n}}$$
$$=abx^{\frac{1}{n}}y^{\frac{1}{n}}$$
$$=ab(xy)^{\frac{1}{n}}$$
$$=ab\sqrt[n]{xy}.$$

Examples. (1) $5\sqrt{3}\times3\sqrt{7}=15\sqrt{21}$.

(2) $2\sqrt{x}\times3\sqrt{x}=6x$.

(3) $\sqrt[4]{a+b}\times\sqrt[4]{a-b}=\sqrt[4]{(a+b)(a-b)}=\sqrt[4]{a^2-b^2}$.

263. If the surds are not in their simplest form, it will save labour to reduce them to this form before multiplication.

Example. The product of $5\sqrt{32}$, $\sqrt{48}$, $2\sqrt{54}$

$$=5\,.\,4\sqrt{2}\times 4\sqrt{3}\times 2\,.\,3\sqrt{6}=480\,.\,\sqrt{2}\,.\,\sqrt{3}\,.\,\sqrt{6}=480\times 6=2880.$$

264. *To multiply surds which are not of the same order: reduce them to equivalent surds of the same order, and proceed as before.*

Example. Multiply $5\sqrt[3]{2}$ by $2\sqrt{5}$.

The product $=5\sqrt[6]{2^2}\times 2\sqrt[6]{5^3}=10\sqrt[6]{2^2\times 5^3}=10\sqrt[6]{500}$.

265. Suppose it is required to find the numerical value of the quotient when $\sqrt{5}$ is divided by $\sqrt{7}$.

At first sight it would seem that we must find the square root of 5, which is 2·236 ..., and then the square root of 7, which is 2·645 ..., and finally divide 2·236 ... by 2·645 ... ; three troublesome operations.

But we may avoid much of this labour by multiplying both numerator and denominator by $\sqrt{7}$, so as to make the denominator a rational quantity. Thus

$$\frac{\sqrt{5}}{\sqrt{7}}=\frac{\sqrt{5}}{\sqrt{7}}\times\frac{\sqrt{7}}{\sqrt{7}}=\frac{\sqrt{5\times 7}}{7}=\frac{\sqrt{35}}{7}.$$

Now $$\sqrt{35}=5\cdot916\ldots$$

$$\therefore\quad \frac{\sqrt{5}}{\sqrt{7}}=\frac{5\cdot916\ldots}{7}=\cdot845\ldots.$$

266. The great utility of this artifice in calculating the numerical value of surd fractions suggests its convenience in the case of *all* surd fractions, even where numerical values are not required. Thus it is usual to simplify $\frac{a\sqrt{b}}{\sqrt{c}}$ as follows:

$$\frac{a\sqrt{b}}{\sqrt{c}}=\frac{a\sqrt{b}\times\sqrt{c}}{\sqrt{c}\times\sqrt{c}}=\frac{a\sqrt{bc}}{c}.$$

The process by which surds are removed from the denominator of any fraction is known as **rationalising the denominator.** It is effected by multiplying both numerator and denominator by any factor which renders the denominator rational. We shall return to this point in Art. 270.

267. The quotient of one surd by another may be found by expressing the result as a fraction, and rationalising the denominator.

Example 1. Divide $4\sqrt{75}$ by $25\sqrt{56}$.

$$\text{The quotient}=\frac{4\sqrt{75}}{25\sqrt{56}}=\frac{4\times5\sqrt{3}}{25\times2\sqrt{14}}=\frac{2\sqrt{3}}{5\sqrt{14}}$$

$$=\frac{2\sqrt{3}\times\sqrt{14}}{5\sqrt{14}\times\sqrt{14}}=\frac{2\sqrt{42}}{5\times14}=\frac{\sqrt{42}}{35}.$$

Example 2. $$\frac{\sqrt[3]{b}}{\sqrt[3]{c^2}}=\frac{\sqrt[3]{b}\times\sqrt[3]{c}}{\sqrt[3]{c^2}\times\sqrt[3]{c}}=\frac{\sqrt[3]{bc}}{\sqrt[3]{c^3}}=\frac{\sqrt[3]{bc}}{c}.$$

EXAMPLES XXXI. c.

Find the value of

1. $2\sqrt{14}\times\sqrt{21}$. **2.** $3\sqrt{8}\times\sqrt{6}$. **3.** $5\sqrt{a}\times2\sqrt{3}$.

4. $2\sqrt{15}\times3\sqrt{5}$. **5.** $8\sqrt{12}\times3\sqrt{24}$. **6.** $\sqrt[3]{x+2}\times\sqrt[3]{x-2}$.

7. $21\sqrt{384}\div8\sqrt{98}$. **8.** $5\sqrt{27}\div3\sqrt{24}$. **9.** $-13\sqrt{125}\div5\sqrt{65}$.

10. $\sqrt[3]{168}\times\sqrt[3]{147}$. **11.** $5\sqrt[3]{128}\times2\sqrt[3]{432}$. **12.** $6\sqrt{14}\div2\sqrt{21}$.

13. $a\sqrt{b^3}\times b^2\sqrt{a}$. **14.** $\dfrac{3\sqrt{11}}{2\sqrt{98}}\div\dfrac{5}{7\sqrt{22}}$. **15.** $\dfrac{3\sqrt{48}}{5\sqrt{112}}\div\dfrac{6\sqrt{84}}{\sqrt{392}}$.

16. $\dfrac{3}{x}\sqrt{\dfrac{a^2}{x}}\times\dfrac{4}{3}\sqrt{\dfrac{x^3}{2a^4}}$. **17.** $\dfrac{3}{a-b}\sqrt{\dfrac{2x}{a-b}}\div\sqrt{\dfrac{18x^3}{(a-b)^5}}$.

Given $\sqrt{2}=1\cdot41421$, $\sqrt{3}=1\cdot73205$, $\sqrt{5}=2\cdot23607$, $\sqrt{6}=2\cdot44949$, $\sqrt{7}=2\cdot64575$: find to four places of decimals the numerical value of

18. $\dfrac{14}{\sqrt{2}}$. **19.** $\dfrac{25}{\sqrt{5}}$. **20.** $\dfrac{10}{\sqrt{7}}$. **21.** $\dfrac{48}{\sqrt{6}}$.

22. $\dfrac{60}{\sqrt{5}}$. **23.** $144\div\sqrt{6}$. **24.** $\sqrt{2}\div\sqrt{3}$. **25.** $\dfrac{1}{2\sqrt{3}}$.

26. $\dfrac{1}{\sqrt{500}}$. **27.** $\dfrac{4}{\sqrt{243}}$. **28.** $\dfrac{25}{\sqrt{252}}$. **29.** $\sqrt{\dfrac{256}{1575}}$.

268. Hitherto we have confined our attention to **simple surds,** such as $\sqrt[4]{5}$, $\sqrt[3]{a}$, $\sqrt{x+y}$. An expression involving two or more simple surds is called a **compound surd;** thus $2\sqrt{a}-3\sqrt{b}$; $\sqrt[3]{a}+\sqrt[4]{b}$ are compound surds.

269. The multiplication of compound surds is performed like the multiplication of compound algebraical expressions.

Example 1. Multiply $2\sqrt{x}-5$ by $3\sqrt{x}$.

$$\text{The product} = 3\sqrt{x}(2\sqrt{x}-5)$$
$$= 6x - 15\sqrt{x}.$$

Example 2. Multiply $2\sqrt{5}+3\sqrt{x}$ by $\sqrt{5}-\sqrt{x}$.

$$\text{The product} = (2\sqrt{5}+3\sqrt{x})(\sqrt{5}-\sqrt{x})$$
$$= 2\sqrt{5}\,.\,\sqrt{5}+3\sqrt{5}\,.\,\sqrt{x}-2\sqrt{5}\,.\,\sqrt{x}-3\sqrt{x}\,.\,\sqrt{x}$$
$$= 10 - 3x + \sqrt{5x}.$$

Example 3. Find the square of $2\sqrt{x}+\sqrt{7-4x}$.

$$(2\sqrt{x}+\sqrt{7-4x})^2 = (2\sqrt{x})^2 + (\sqrt{7-4x})^2 + 4\sqrt{x}\,.\,\sqrt{7-4x}$$
$$= 4x + 7 - 4x + 4\sqrt{7x-4x^2}$$
$$= 7 + 4\sqrt{7x-4x^2}.$$

EXAMPLES XXXI. d.

Find the value of

1. $(3\sqrt{x}-5)\times 2\sqrt{x}$.
2. $(\sqrt{x}-\sqrt{a})\times 2\sqrt{x}$.
3. $(\sqrt{a}+\sqrt{b})\times\sqrt{ab}$.
4. $(\sqrt{x+y}-1)\times\sqrt{x+y}$.
5. $(2\sqrt{3}+3\sqrt{2})^2$.
6. $(\sqrt{7}+5\sqrt{3})(2\sqrt{7}-4\sqrt{3})$.
7. $(3\sqrt{5}-4\sqrt{2})(2\sqrt{5}+3\sqrt{2})$.
8. $(3\sqrt{a}-2\sqrt{x})(2\sqrt{a}+3\sqrt{x})$.
9. $(\sqrt{x}+\sqrt{x-1})\times\sqrt{x-1}$.
10. $(\sqrt{x+a}-\sqrt{x-a})\times\sqrt{x+a}$.
11. $(\sqrt{a+x}-2\sqrt{a})^2$.
12. $(2\sqrt{a}-\sqrt{1+4a})^2$.
13. $(\sqrt{a+x}-\sqrt{a-x})^2$.
14. $(\sqrt{a+x}-2)(\sqrt{a+x}-1)$.
15. $(\sqrt{2}+\sqrt{3}-\sqrt{5})(\sqrt{2}+\sqrt{3}+\sqrt{5})$.
16. $(\sqrt{5}+3\sqrt{2}+\sqrt{7})(\sqrt{5}+3\sqrt{2}-\sqrt{7})$.

Write down the square of

17. $\sqrt{2x+a}-\sqrt{2x-a}$.
18. $\sqrt{x^2-2y^2}+\sqrt{x^2+2y^2}$.
19. $\sqrt{m+n}+\sqrt{m-n}$.
20. $3\sqrt{a^2+b^2}-2\sqrt{a^2-b^2}$.
21. $3x\sqrt{2}-3\sqrt{7-2x^2}$.
22. $\sqrt{4x^2+1}-\sqrt{4x^2-1}$.

270. One case of the multiplication of compound surds deserves careful attention. For if we multiply together the sum and the difference of any two quadratic surds we obtain a rational product.

Examples. (1) $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})=(\sqrt{a})^2-(\sqrt{b})^2=a-b.$

(2) $(3\sqrt{5}+4\sqrt{3})(3\sqrt{5}-4\sqrt{3})=(3\sqrt{5})^2-(4\sqrt{3})^2=45-48=-3.$

Similarly, $(4-\sqrt{a+b})(4+\sqrt{a+b})=(4)^2-(\sqrt{a+b})^2=16-a-b.$

271. Definition. When two binomial quadratic surds differ only in the sign which connects their terms they are said to be **conjugate.**

Thus $3\sqrt{7}+5\sqrt{11}$ is conjugate to $3\sqrt{7}-5\sqrt{11}$.

Similarly, $a-\sqrt{a^2-x^2}$ is conjugate to $a+\sqrt{a^2-x^2}$.

The product of two conjugate surds is rational. [Art. 270.]

Example. $(3\sqrt{a}+\sqrt{x-9a})(3\sqrt{a}-\sqrt{x-9a})$

$$=(3\sqrt{a})^2-(\sqrt{x-9a})^2=9a-(x-9a)=18a-x.$$

272. The only case of the division of compound surds which we shall here consider is that in which the divisor is a binomial quadratic surd. If we express the division by means of a fraction, we can always rationalise the denominator by multiplying numerator and denominator by the surd which is conjugate to the divisor.

Example 1. Divide $4+3\sqrt{2}$ by $5-3\sqrt{2}$.

$$\text{The quotient}=\frac{4+3\sqrt{2}}{5-3\sqrt{2}}=\frac{4+3\sqrt{2}}{5-3\sqrt{2}}\times\frac{5+3\sqrt{2}}{5+3\sqrt{2}}$$

$$=\frac{20+18+12\sqrt{2}+15\sqrt{2}}{25-18}=\frac{38+27\sqrt{2}}{7}.$$

Example 2. Rationalise the denominator of $\dfrac{b^2}{\sqrt{a^2+b^2}+a}$.

$$\text{The expression}=\frac{b^2}{\sqrt{a^2+b^2}+a}\times\frac{\sqrt{a^2+b^2}-a}{\sqrt{a^2+b^2}-a}$$

$$=\frac{b^2\{\sqrt{a^2+b^2}-a\}}{(a^2+b^2)-a^2}$$

$$=\sqrt{a^2+b^2}-a.$$

Example 3. Divide $\dfrac{\sqrt{3}+\sqrt{2}}{2-\sqrt{3}}$ by $\dfrac{7+4\sqrt{3}}{\sqrt{3}-\sqrt{2}}$.

$$\text{The quotient}=\frac{\sqrt{3}+\sqrt{2}}{2-\sqrt{3}}\times\frac{\sqrt{3}-\sqrt{2}}{7+4\sqrt{3}}=\frac{(\sqrt{3})^2-(\sqrt{2})^2}{14-12+8\sqrt{3}-7\sqrt{3}}$$

$$=\frac{1}{2+\sqrt{3}}=2-\sqrt{3}, \text{ on rationalising.}$$

Example 4. Given $\sqrt{5}=2{\cdot}236068$, find the value of $\dfrac{87}{7-2\sqrt{5}}$.

Rationalising the denominator,

$$\frac{87}{7-2\sqrt{5}}=\frac{87(7+2\sqrt{5})}{49-20}=3(7+2\sqrt{5})=34{\cdot}416408.$$

It will be seen that by rationalising the denominator we have avoided the use of a divisor consisting of 7 figures.

EXAMPLES XXXI. e.

Find the value of

1. $(9\sqrt{2}-7)(9\sqrt{2}+7)$.

2. $(3+5\sqrt{7})(3-5\sqrt{7})$.

3. $(5\sqrt{8}-2\sqrt{7})(5\sqrt{8}+2\sqrt{7})$.

4. $(2\sqrt{11}+5\sqrt{2})(2\sqrt{11}-5\sqrt{2})$.

5. $(\sqrt{a}+2\sqrt{b})(\sqrt{a}-2\sqrt{b})$.

6. $(3c-2\sqrt{x})(3c+2\sqrt{x})$.

7. $(\sqrt{a+x}-\sqrt{a})(\sqrt{a+x}+\sqrt{a})$.

8. $(\sqrt{2p+3q}-2\sqrt{q})(\sqrt{2p+3q}+2\sqrt{q})$.

9. $(\sqrt{a+x}+\sqrt{a-x})(\sqrt{a+x}-\sqrt{a-x})$.

10. $(5\sqrt{x^2-3y^2}+7a)(5\sqrt{x^2-3y^2}-7a)$.

11. $29\div(11+3\sqrt{7})$.

12. $17\div(3\sqrt{7}+2\sqrt{3})$.

13. $(3\sqrt{2}-1)\div(3\sqrt{2}+1)$.

14. $(2\sqrt{3}+7\sqrt{2})\div(5\sqrt{3}-4\sqrt{2})$.

15. $(2x-\sqrt{xy})\div(2\sqrt{xy}-y)$.

16. $(3+\sqrt{5})(\sqrt{5}-2)\div(5-\sqrt{5})$.

17. $\dfrac{\sqrt{a}}{\sqrt{a}-\sqrt{x}}\div\dfrac{\sqrt{a}+\sqrt{x}}{\sqrt{x}}$.

18. $\dfrac{2\sqrt{15}+8}{5+\sqrt{15}}\div\dfrac{8\sqrt{3}-6\sqrt{5}}{5\sqrt{3}-3\sqrt{5}}$.

Rationalise the denominator of

19. $\dfrac{25\sqrt{3}-4\sqrt{2}}{7\sqrt{3}-5\sqrt{2}}$.

20. $\dfrac{10\sqrt{6}-2\sqrt{7}}{3\sqrt{6}+2\sqrt{7}}$.

21. $\dfrac{\sqrt{7}+\sqrt{2}}{9+2\sqrt{14}}$.

22. $\dfrac{2\sqrt{3}+3\sqrt{2}}{5+2\sqrt{6}}$.

23. $\dfrac{y^2}{x+\sqrt{x^2-y^2}}$.

24. $\dfrac{x^2}{\sqrt{x^2+a^2}+a}$.

25. $\dfrac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}$.

26. $\dfrac{2\sqrt{a+b}+3\sqrt{a-b}}{2\sqrt{a+b}-\sqrt{a-b}}$.

27. $\dfrac{\sqrt{9+x^2}-3}{\sqrt{9+x^2}+3}$.

28. $\dfrac{3+\sqrt{6}}{5\sqrt{3}-2\sqrt{12}-\sqrt{32}+\sqrt{50}}$.

Given $\sqrt{2}=1{\cdot}41421$, $\sqrt{3}=1{\cdot}73205$, $\sqrt{5}=2{\cdot}23607$: find to four places of decimals the value of

29. $\dfrac{1}{2+\sqrt{3}}$.

30. $\dfrac{3+\sqrt{5}}{\sqrt{5}-2}$.

31. $\dfrac{\sqrt{5}+\sqrt{3}}{4+\sqrt{15}}$.

32. $\dfrac{\sqrt{5}-2}{9-4\sqrt{5}}$.

33. $\dfrac{7\sqrt{5}+15}{\sqrt{5}-1}\times\dfrac{\sqrt{5}-2}{3+\sqrt{5}}$.

34. $(2-\sqrt{3})(7-4\sqrt{3})\div(3\sqrt{3}-5)$.

273. *The square root of a rational quantity cannot be partly rational and partly a quadratic surd.*

If possible let $\sqrt{n}=a+\sqrt{m}$;

then by squaring, $n=a^2+m+2a\sqrt{m}$;

$$\therefore \quad \sqrt{m}=\frac{n-a^2-m}{2a};$$

that is a surd is equal to a rational quantity; which is impossible.

274. *If* $x+\sqrt{y}=a+\sqrt{b}$, *where* x *and* a *are both rational and* $\sqrt{y}$ *and* $\sqrt{b}$ *are both irrational, then will* $x=a$ *and* $y=b$.

For if x is not equal to a, let $x=a+m$; then

$$a+m+\sqrt{y}=a+\sqrt{b};$$

that is, $\sqrt{b}=m+\sqrt{y}$;

which is impossible. [Art. 273.]

Therefore $x=a$,

and consequently, $y=b$.

If therefore $x+\sqrt{y}=a+\sqrt{b}$,

we must also have $x-\sqrt{y}=a-\sqrt{b}$.

275. It appears from the preceding article that in any equation of the form

$$X+\sqrt{Y}=A+\sqrt{B}\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

we may equate the rational parts on each side, and also the irrational parts; so that the equation (1) is really equivalent to *two* independent equations, $X=A$ and $Y=B$. *But this is only true when* $\sqrt{Y}$ *and* $\sqrt{B}$ *are irrational.*

276. *If* $\sqrt{a+\sqrt{b}}=\sqrt{x}+\sqrt{y}$ *then will* $\sqrt{a-\sqrt{b}}=\sqrt{x}-\sqrt{y}$.

For by squaring, we obtain

$$a+\sqrt{b}=x+2\sqrt{xy}+y\,;$$

$$\therefore \quad a=x+y, \ \sqrt{b}=2\sqrt{xy}. \qquad \text{[Art. 275.]}$$

Hence $$a-\sqrt{b}=x-2\sqrt{xy}+y,$$

and $$\sqrt{a-\sqrt{b}}=\sqrt{x}-\sqrt{y}.$$

277. *To find the square root of* $a+\sqrt{b}$.

Suppose $$\sqrt{a+\sqrt{b}}=\sqrt{x}+\sqrt{y}\,;$$

then as in the last article,

$$x+y=a \ \ldots\ldots\ldots\ldots(1),$$

$$2\sqrt{xy}=\sqrt{b} \ \ldots\ldots\ldots\ldots(2).$$

$$\therefore \quad (x-y)^2=(x+y)^2-4xy$$

$$=a^2-b, \qquad \text{from (1) and (2)}$$

$$\therefore \quad x-y=\sqrt{a^2-b}.$$

Combining this with (1) we find

$$x=\frac{a+\sqrt{a^2-b}}{2}, \text{ and } y=\frac{a-\sqrt{a^2-b}}{2}$$

$$\therefore \quad \sqrt{a+\sqrt{b}}=\sqrt{\frac{a+\sqrt{(a^2-b)}}{2}}+\sqrt{\frac{a-\sqrt{(a^2-b)}}{2}}.$$

278. From the values just found for x and y, it appears that each of them is itself a compound surd unless a^2-b is a perfect square. Hence the method of Art. 277 for finding the square root of $a+\sqrt{b}$ is of no practical utility except when a^2-b is a perfect square.

Example. Find the square root of $16+2\sqrt{55}$.

Assume $$\sqrt{16+2\sqrt{55}}=\sqrt{x}+\sqrt{y}.$$

Then $$16+2\sqrt{55}=x+2\sqrt{xy}+y.$$

$$\therefore \quad x+y=16 \ \ldots\ldots\ldots\ldots(1),$$

$$2\sqrt{xy}=2\sqrt{55} \ \ldots\ldots\ldots\ldots(2).$$

$$\therefore \quad (x-y)^2=(x+y)^2-4xy$$

$$=16^2-4\times 55, \qquad \text{by (1) and (2).}$$

$$=4\times 9.$$

$$\therefore \quad x-y=\pm 6 \ \ldots\ldots\ldots\ldots(3).$$

From (1) and (3) we obtain

$$x=11, \text{ or } 5, \text{ and } y=5, \text{ or } 11.$$

That is, the required square root is $\sqrt{11}+\sqrt{5}$.

In the same way we may shew that

$$\sqrt{16-2\sqrt{55}}=\sqrt{11}-\sqrt{5}.$$

Note. Since every quantity has two square roots equal in magnitude but opposite in sign, strictly speaking we should have

$$\text{the square root of } 16+2\sqrt{55}=\pm(\sqrt{11}+\sqrt{5}),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 16-2\sqrt{55}=\pm(\sqrt{11}-\sqrt{5}).$$

However it is usually sufficient to take the positive value of the square root, so that in assuming $\sqrt{a-\sqrt{b}}=\sqrt{x}-\sqrt{y}$ it is understood that x is greater than y. With this proviso it will be unnecessary in any numerical example to use the double sign at the stage of work corresponding to equation (3) of the last example.

279. When the binomial whose square root we are seeking consists of *two* quadratic surds, we proceed as explained in the following example.

Example. Find the square root of $\sqrt{175}-\sqrt{147}$.

Since $\sqrt{175}-\sqrt{147}=\sqrt{7}(\sqrt{25}-\sqrt{21})=\sqrt{7}(5-\sqrt{21})$.

$$\therefore \sqrt{\sqrt{175}-\sqrt{147}}=\sqrt[4]{7}\,.\,\sqrt{5-\sqrt{21}}.$$

And, proceeding as in the last article,

$$\sqrt{5-\sqrt{21}}=\sqrt{\frac{7}{2}}-\sqrt{\frac{3}{2}};$$

$$\therefore \sqrt{\sqrt{175}-\sqrt{147}}=\sqrt[4]{7}\left(\sqrt{\frac{7}{2}}-\sqrt{\frac{3}{2}}\right).$$

280. The square root of a binomial surd may often be found by inspection.

Example 1. Find the square root of $11+2\sqrt{30}$.

We have only to find two quantities whose sum is 11, and whose product is 30; thus

$$11+2\sqrt{30}=6+5+2\sqrt{6\times 5}$$
$$=(\sqrt{6}+\sqrt{5})^2.$$
$$\therefore \sqrt{11+2\sqrt{30}}=\sqrt{6}+\sqrt{5}.$$

Example 2. Find the square root of $53-12\sqrt{10}$.

First write the binomial so that the surd part has a coefficient 2; thus $\qquad 53-12\sqrt{10}=53-2\sqrt{360}$.

We have now to find two quantities whose sum is 53 and whose product is 360; these are 45 and 8;

hence
$$53-12\sqrt{10}=45+8-2\sqrt{45\times 8}$$
$$=(\sqrt{45}-\sqrt{8})^2;$$
$$\therefore\ \sqrt{53-12\sqrt{10}}=\sqrt{45}-\sqrt{8}$$
$$=3\sqrt{5}-2\sqrt{2}.$$

EXAMPLES XXXI. f.

Find the square root of each of the following binomial surds:

1. $7-2\sqrt{10}$. 2. $13+2\sqrt{30}$. 3. $8-2\sqrt{7}$.

4. $5+2\sqrt{6}$. 5. $75+12\sqrt{21}$. 6. $18-8\sqrt{5}$.

7. $41-24\sqrt{2}$. 8. $83+12\sqrt{35}$. 9. $47-4\sqrt{33}$.

10. $2\frac{1}{4}+\sqrt{5}$. 11. $4\frac{1}{3}-\frac{4}{3}\sqrt{3}$. 12. $16+5\sqrt{7}$.

13. $\sqrt{27}+2\sqrt{6}$. 14. $\sqrt{32}-\sqrt{24}$. 15. $3\sqrt{5}+\sqrt{40}$.

Find the fourth roots of the following binomial surds:

16. $17+12\sqrt{2}$. 17. $56+24\sqrt{5}$. 18. $\frac{3}{2}\sqrt{5}+3\frac{1}{2}$.

19. $14+8\sqrt{3}$. 20. $49-20\sqrt{6}$. 21. $248+32\sqrt{60}$.

Find, by inspection, the value of

22. $\sqrt{3-2\sqrt{2}}$. 23. $\sqrt{4+2\sqrt{3}}$. 24. $\sqrt{6-2\sqrt{5}}$.

25. $\sqrt{19+8\sqrt{3}}$. 26. $\sqrt{8+2\sqrt{15}}$. 27. $\sqrt{9-2\sqrt{14}}$.

28. $\sqrt{11+4\sqrt{6}}$. 29. $\sqrt{15-4\sqrt{14}}$. 30. $\sqrt{29+6\sqrt{22}}$.

Equations involving Surds.

281. Sometimes equations are proposed in which the unknown quantity appears under the radical sign. Such equations are very varied in character and often require special artifices for their solution. Here we shall only consider a few of the simpler cases, which can generally be solved by the following method. Bring to one side of the equation a single radical term by itself: on squaring both sides this radical will disappear. By repeating this process any remaining radicals can in turn be removed.

Example 1. Solve $2\sqrt{x}-\sqrt{4x-11}=1.$

Transposing $2\sqrt{x}-1=\sqrt{4x-11}.$

Square both sides; then $4x-4\sqrt{x}+1=4x-11,$

$$4\sqrt{x}=12,$$

$$\sqrt{x}=3;$$

$$\therefore \quad x=9.$$

Example 2. Solve $2+\sqrt[3]{x-5}=13.$

Transposing $\sqrt[3]{x-5}=11.$

Here we must *cube* both sides; thus $x-5=1331$;

whence $x=1336.$

Example 3. Solve $\sqrt{x+5}+\sqrt{3x+4}=\sqrt{12x+1}.$

Squaring both sides,

$$x+5+3x+4+2\sqrt{(x+5)(3x+4)}=12x+1.$$

Transposing and dividing by 2,

$$\sqrt{(x+5)(3x+4)}=4x-4 \quad\ldots\ldots\ldots\ldots\ldots(1).$$

Squaring, $(x+5)(3x+4)=16x^2-32x+16,$

or $13x^2-51x-4=0,$

$$(x-4)(13x+1)=0;$$

$$\therefore \quad x=4, \text{ or } -\frac{1}{13}.$$

If we proceed to verify the solution by substituting these values in the original equation, it will be found that it is satisfied by $x=4$, but not by $x=-\frac{1}{13}$. But this latter value will be found on trial to satisfy the given equation if we alter the sign of the second radical; thus

$$\sqrt{x+5}-\sqrt{3x+4}=\sqrt{12x+1}.$$

On squaring this and reducing, we obtain

$$-\sqrt{(x+5)(3x+4)}=4x-4 \quad\ldots\ldots\ldots\ldots\ldots\ldots(2);$$

and a comparison of (1) and (2) shews that in the next stage of the work *the same quadratic equation is obtained* in each case, the roots of which are 4 and $-\frac{1}{13}$, as already found.

From this it appears that when the solution of an equation requires that both sides should be squared, we cannot be certain without trial which of the values found for the unknown quantity will satisfy the original equation.

In order that all the values found by the solution of the equation may be applicable it will be necessary to take into account both signs of the radicals in the given equation.

EXAMPLES XXXI. g.

Solve the equations :

1. $\sqrt{x-5}=3.$
2. $\sqrt[3]{4x-7}=5.$
3. $7-\sqrt{x-4}=3.$
4. $13-\sqrt[3]{5x-4}=7.$
5. $\sqrt{5x-1}=2\sqrt{x+3}.$
6. $2\sqrt{3-7x}-3\sqrt{8x-12}=0.$
7. $2\sqrt[3]{5x-35}=5\sqrt[3]{2x-7}.$
8. $\sqrt{9x^2-11x-5}=3x-2.$
9. $\sqrt[4]{2x+11}=\sqrt{5}.$
10. $\sqrt{4x^2-7x+1}=2x-1\frac{4}{5}.$
11. $\sqrt{x+25}=1+\sqrt{x}.$
12. $\sqrt{8x+33}-3=2\sqrt{2x}.$
13. $\sqrt{x+3}+\sqrt{x}=5.$
14. $10-\sqrt{25+9x}=3\sqrt{x}.$
15. $\sqrt{x-4}+3=\sqrt{x+11}.$
16. $\sqrt{9x-8}=3\sqrt{x+4}-2.$
17. $\sqrt{4x+5}-\sqrt{x}=\sqrt{x+3}.$
18. $\sqrt{25x-29}-\sqrt{4x-11}=3\sqrt{x}.$
19. $\sqrt{8x+17}-\sqrt{2x}=\sqrt{2x+9}.$
20. $\sqrt{3x-11}+\sqrt{3x}=\sqrt{12x-23}.$
21. $\sqrt{12x-5}-\sqrt{3x-1}=\sqrt{27x-2}.$
22. $\sqrt{x+3}+\sqrt{x+8}-\sqrt{4x+21}=0.$
23. $\sqrt{x+2}+\sqrt{4x+1}-\sqrt{9x+7}=0.$
24. $\sqrt{x+4ab}=2a+\sqrt{x}.$
25. $\sqrt{x}+\sqrt{4a+x}=2\sqrt{b+x}.$
26. $\sqrt{a-x}+\sqrt{b+x}=\sqrt{a}+\sqrt{b}.$
27. $5\sqrt[3]{70x+29}=9\sqrt[3]{14x-15}.$
28. $\sqrt[3]{x^3-3x^2+7x-11}=x-1.$
29. $\sqrt[3]{8x^3+12x^2+12x-11}=2x+1.$
30. $\sqrt[3]{1+x}+\sqrt[3]{1-x}=\sqrt[3]{2}.$

282. When radicals appear in a fractional form in an equation, we must clear of fractions in the ordinary way, combining the irrational factors by the rules already explained in this chapter.

Example 1. Solve $\dfrac{6\sqrt{x}-11}{3\sqrt{x}}=\dfrac{2\sqrt{x}+1}{\sqrt{x}+6}.$

Multiplying across, we have

$$6x+25\sqrt{x}-66=6x+3\sqrt{x},$$

that is,

$$25\sqrt{x}-3\sqrt{x}=66,$$

$$22\sqrt{x}=66,$$

$$\sqrt{x}=3,$$

$$x=9.$$

Example 2. Solve $\sqrt{9+2x}-\sqrt{2x}=\dfrac{5}{\sqrt{9+2x}}$.

Clearing of fractions, $9+2x-\sqrt{2x(9+2x)}=5,$

$$4+2x=\sqrt{2x(9+2x)}.$$

Squaring, $16+16x+4x^2=18x+4x^2,$

$$16=2x,$$

$$x=8.$$

EXAMPLES XXXI. h.

Solve the equations:

1. $\dfrac{6\sqrt{x}-21}{3\sqrt{x}-14}=\dfrac{8\sqrt{x}-11}{4\sqrt{x}-13}$.

2. $\dfrac{9\sqrt{x}-23}{3\sqrt{x}-8}=\dfrac{6\sqrt{x}-17}{2\sqrt{x}-6}$.

3. $\dfrac{\sqrt{x}+3}{\sqrt{x}-2}=\dfrac{3\sqrt{x}-5}{3\sqrt{x}-13}$.

4. $2-\dfrac{\sqrt{x}+3}{\sqrt{x}+2}=\dfrac{\sqrt{x}+9}{\sqrt{x}+7}$.

5. $\dfrac{2\sqrt{x}-1}{2\sqrt{x}+\frac{4}{3}}=\dfrac{\sqrt{x}-2}{\sqrt{x}-\frac{4}{3}}$.

6. $\dfrac{6\sqrt{x}-7}{\sqrt{x}-1}-5=\dfrac{7\sqrt{x}-26}{7\sqrt{x}-21}$.

7. $\dfrac{12\sqrt{x}-11}{4\sqrt{x}-4\frac{2}{3}}=\dfrac{6\sqrt{x}+5}{2\sqrt{x}+\frac{2}{3}}$.

8. $\sqrt{1+x}+\sqrt{x}=\dfrac{2}{\sqrt{1+x}}$.

9. $\sqrt{x-1}+\sqrt{x}=\dfrac{2}{\sqrt{x}}$.

10. $\sqrt{x}-\sqrt{x-8}=\dfrac{2}{\sqrt{x-8}}$.

11. $\sqrt{x+5}+\sqrt{x}=\dfrac{10}{\sqrt{x}}$.

12. $2\sqrt{x}-\sqrt{4x-3}=\dfrac{1}{\sqrt{4x-3}}$.

13. $3\sqrt{x}=\dfrac{8}{\sqrt{9x-32}}+\sqrt{9x-32}$.

14. $\sqrt{x}-7=\dfrac{1}{\sqrt{x}+7}$.

15. $(\sqrt{x}+11)(\sqrt{x}-11)+110=0$.

16. $2\sqrt{x}=\dfrac{12-6\sqrt{x}}{2\sqrt{x}-3}$.

17. $3\sqrt{x}-1=\dfrac{5}{3\sqrt{x}+7}+6$.

18. $\dfrac{x-1}{\sqrt{x}-1}=3+\dfrac{\sqrt{x}+1}{2}$.

19. $\dfrac{1}{1-x}+\dfrac{1}{\sqrt{x}+1}+\dfrac{1}{\sqrt{x}-1}=0$.

20. $2=\dfrac{\sqrt{2+x}+\sqrt{2-x}}{\sqrt{2+x}-\sqrt{2-x}}$.

21. $\dfrac{2x-3}{\sqrt{x-2}+1}=2\sqrt{x-2}-1$.

22. $\dfrac{2}{x-6+\sqrt{x}}+\dfrac{3}{\sqrt{x}-2}=\dfrac{4}{\sqrt{x}+3}$.

EXAMPLES XXXI. i.

1. $\dfrac{a+x}{\sqrt{a}+\sqrt{x-a}}+\dfrac{a-x}{\sqrt{a}+\sqrt{x-a}}=\sqrt{a}.$

2. $\dfrac{\sqrt{x}+3}{\sqrt{x}-8}=\dfrac{\sqrt{x}-18}{\sqrt{x}+4}.$

3. $\dfrac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}=c.$

4. $\dfrac{\sqrt{x}-1}{x-1}=1-\dfrac{\sqrt{x}}{2\sqrt{x}-5}.$

5. $\dfrac{x+\sqrt{x^2-1}}{x-\sqrt{x^2-1}}+\dfrac{x-\sqrt{x^2-1}}{x+\sqrt{x^2-1}}=4x(x-1).$

6. $\sqrt{\dfrac{x+1}{x-1}}+\sqrt{\dfrac{x-1}{x+1}}=a.$

7. $\dfrac{\sqrt{x^2+1}-\sqrt{x^2-1}}{\sqrt{x^2+1}+\sqrt{x^2-1}}=\dfrac{1}{2}.$

8. $\dfrac{\sqrt{3x^2+4}+2}{\sqrt{3x^2+4}-2}=3.$

9. $\sqrt{\dfrac{x+3}{x-3}}+\sqrt{\dfrac{x-3}{x+3}}=5.$

10. $\dfrac{2}{x+\sqrt{2-x^2}}+\dfrac{2}{x-\sqrt{2-x^2}}=x.$

11. $\dfrac{\sqrt{1+x}+\sqrt{x-7}}{\sqrt{1+x}-\sqrt{x-7}}=2.$

12. $\sqrt{\dfrac{x}{a}}+\sqrt{\dfrac{a}{x}}=\sqrt{\dfrac{x}{b}}+\sqrt{\dfrac{b}{x}}.$

13. $\dfrac{1}{x+\sqrt{x^2-50}}+\dfrac{1}{x-\sqrt{x^2-50}}=\dfrac{7}{25}.$

14. $\sqrt{4+x}+\sqrt{6-x}=\sqrt{6+2x}.$

15. $x-\sqrt{x^2-1}=\dfrac{1}{2(x+1)}.$

CHAPTER XXXII.

THE THEORY OF QUADRATIC EQUATIONS.

283. IN Chapter XXV. it was shewn that after suitable reduction every quadratic equation may be written in the form

$$ax^2+bx+c=0 \qquad\qquad (1),$$

and that the solution of the equation is

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \qquad\qquad (2).$$

We shall now prove some important propositions connected with the roots and coefficients of all equations of which (1) is the type.

284. *A quadratic equation cannot have more than two roots.*

For, if possible, let the equation $ax^2+bx+c=0$ have three *different* roots α, β, γ. Then since each of these values must satisfy the equation, we have

$$a\alpha^2+b\alpha+c=0 \qquad\qquad (1),$$

$$a\beta^2+b\beta+c=0 \qquad\qquad (2),$$

$$a\gamma^2+b\gamma+c=0 \qquad\qquad (3),$$

From (1) and (2), by subtraction,

$$a(\alpha^2-\beta^2)+b(\alpha-\beta)=0\,;$$

divide out by $\alpha-\beta$ which, by hypothesis, is not zero; then

$$a(\alpha+\beta)+b=0.$$

Similarly from (2) and (3)

$$a(\beta+\gamma)+b=0\,;$$

$\therefore$ by subtraction $\qquad a(\alpha-\gamma)=0\,;$

which is impossible, since, by hypothesis, a is not zero, and α is not equal to γ. Hence there cannot be three different roots.

285. The terms 'unreal', 'imaginary', and 'impossible' are all used in the same sense: namely, to denote expressions which involve the square root of a negative quantity. It is important that the student should clearly distinguish between the terms *real* and *rational*, *imaginary* and *irrational*. Thus $\sqrt{25}$ or 5, $3\frac{1}{2}$, $-\frac{5}{8}$ are rational and real; $\sqrt{7}$ is irrational but real; while $\sqrt{-7}$ is irrational and also imaginary.

286. In Art. 283 if the two roots in (2) are denoted by α and β, we have

$$\alpha=\frac{-b+\sqrt{b^2-4ac}}{2a}, \qquad \beta=\frac{-b-\sqrt{b^2-4ac}}{2a}.$$

(1) If b^2-4ac, the quantity under the radical, is positive, α and β are real and unequal.

(2) If b^2-4ac is zero, α and β are real and equal, each reducing in this case to $-\dfrac{b}{2a}$.

(3) If b^2-4ac is negative, α and β are imaginary and unequal.

(4) If b^2-4ac is a perfect square, α and β are rational and unequal.

By applying these tests the nature of the roots of any quadratic may be determined without solving the equation.

Example 1. Shew that the equation $2x^2-6x+7=0$ cannot be satisfied by any real values of x.

Here $a=2$, $b=-6$, $c=7$; so that

$$b^2-4ac=(-6)^2-4\,.\,2\,.\,7=-20.$$

Therefore the roots are imaginary.

Note. If the equation is solved graphically as in Art. 330, it will be found that the graph does not cut the axis of x. Thus there are no real values of x which make $2x^2-6x+7$ equal to zero.

Example 2. For what value of k will the equation $3x^2-6x+k=0$ have equal roots?

The condition for equal roots gives

$$(-6)^2-4\,.\,3\,.\,k=0,$$

whence $$k=3.$$

Example 3. Shew that the roots of the equation

$$x^2-2px+p^2-q^2+2qr-r^2=0$$

are rational.

The roots will be rational provided $(-2p)^2-4(p^2-q^2+2qr-r^2)$ is a perfect square. But this expression reduces to $4(q^2-2qr+r^2)$, or $4(q-r)^2$. Hence the roots are rational.

287. Since $\alpha = \dfrac{-b+\sqrt{b^2-4ac}}{2a}$, $\beta = \dfrac{-b-\sqrt{b^2-4ac}}{2a}$,

we have by addition

$$\alpha+\beta = \frac{-b+\sqrt{b^2-4ac}-b-\sqrt{b^2-4ac}}{2a},$$

$$= -\frac{2b}{2a} = -\frac{b}{a} \quad \text{.........................} (1);$$

and by multiplication we have

$$\alpha\beta = \frac{(-b+\sqrt{b^2-4ac})(-b-\sqrt{b^2-4ac})}{4a^2}$$

$$= \frac{(-b)^2-(b^2-4ac)}{4a^2}$$

$$= \frac{4ac}{4a^2} = \frac{c}{a} \quad \text{.........................} (2).$$

By writing the equation in the form

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0,$$

these results may also be expressed as follows :

In a quadratic equation *where the coefficient of the first term is unity,*

(i) the sum of the roots is equal to the coefficient of x with its sign changed ;

(ii) the product of the roots is equal to the third term.

Note. In any equation the term which does not contain the unknown quantity is frequently called *the absolute term.*

288. Since $\quad -\dfrac{b}{a} = \alpha+\beta$, and $\dfrac{c}{a} = \alpha\beta$,

the equation $x^2 + \dfrac{b}{a}x + \dfrac{c}{a}$ may be written

$$x^2 - (\alpha+\beta)x + \alpha\beta = 0 \quad \text{.........................} (1).$$

Hence any quadratic may also be expressed in the form

$$x^2 - (\text{sum of roots})\, x + \text{product of roots} = 0 \quad \text{............} (2)$$

Again, from (1) we have

$$(x-\alpha)(x-\beta) = 0 \quad \text{.........................} (3).$$

We may now easily form an equation with given roots.

Example 1. Form the equation whose roots are 3 and -2.

The equation is $$(x-3)(x+2)=0,$$
or $$x^2-x-6=0.$$

Example 2. Form the equation whose roots are $\frac{3}{7}$ and $-\frac{4}{5}$.

The equation is $$\left(x-\frac{3}{7}\right)\left(x+\frac{4}{5}\right)=0;$$
that is, $$(7x-3)(5x+4)=0,$$
or $$35x^2+13x-12=0.$$

When the roots are irrational it is easier to use the following method :

Example 3. Form the equation whose roots are $2+\sqrt{3}$ and $2-\sqrt{3}$.

We have $$\text{sum of roots}=4,$$
$$\text{product of roots}=1;$$
$\therefore$ the equation is $$x^2-4x+1=0,$$
by using formula (2) of the present article.

289. The results of Art. 287 are most important, and they are generally sufficient to solve problems connected with the roots of quadratics. In such questions *the roots should never be considered singly*, but use should be made of the relations obtained by writing down the sum of the roots, and their product, in terms of the coefficients of the equation.

Example 1. If α and β are the roots of $x^2-px+q=0$, find the value of (1) $\alpha^2+\beta^2$, (2) $\alpha^3+\beta^3$.

We have $$\alpha+\beta=p,$$
$$\alpha\beta=q.$$
$$\therefore\quad \alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$$
$$=p^2-2q.$$
Again, $$\alpha^3+\beta^3=(\alpha+\beta)(\alpha^2+\beta^2-\alpha\beta)$$
$$=p\{(\alpha+\beta)^2-3\alpha\beta\}$$
$$=p(p^2-3q).$$

Example 2. If α, β are the roots of the equation $lx^2+mx+n=0$, find the equation whose roots are $\frac{\alpha}{\beta}$, $\frac{\beta}{\alpha}$.

We have $$\text{sum of roots}=\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{\alpha^2+\beta^2}{\alpha\beta},$$
$$\text{product of roots}=\frac{\alpha}{\beta}\cdot\frac{\beta}{\alpha}=1;$$

$\therefore$ by Art. 288 the required equation is

$$x^2 - \left(\frac{\alpha^2+\beta^2}{\alpha\beta}\right)x + 1 = 0,$$

or $$\alpha\beta x^2 - (\alpha^2+\beta^2)x + \alpha\beta = 0.$$

As in the last example $\alpha^2+\beta^2 = \frac{m^2-2nl}{l^2}$, and $\alpha\beta = \frac{n}{l}$.

$\therefore$ the equation is $$\frac{n}{l}x^2 - \frac{m^2-2nl}{l^2}x + \frac{n}{l} = 0,$$

or $$nlx^2 - (m^2-2nl)x + nl = 0.$$

Example 3. Find the condition that the roots of the equation $ax^2+bx+c=0$ should be (1) equal in magnitude and opposite in sign, (2) reciprocals.

The roots will be equal in magnitude and opposite in sign if their sum is zero; therefore $-\frac{b}{a}=0$, or $b=0$.

Again, the roots will be reciprocals when their product is unity; therefore $\frac{c}{a}=1$, or $c=a$.

Example 4. Find the relation which must subsist between the coefficients of the equation $px^2+qx+r=0$, when one root is three times the other.

We have $$\alpha+\beta = -\frac{q}{p}, \quad \alpha\beta = \frac{r}{p};$$

but since $\alpha = 3\beta$, we obtain by substitution

$$4\beta = -\frac{q}{p}, \quad 3\beta^2 = \frac{r}{p}.$$

From the first of these equations $\beta^2 = \frac{q^2}{16p^2}$, and from the second $\beta^2 = \frac{r}{3p}$.

$$\therefore \frac{q^2}{16p^2} = \frac{r}{3p},$$

or $$3q^2 = 16pr,$$

which is the required condition.

EXAMPLES XXXII.

Find (without actual solution) the nature of the roots of the following equations :

1. $x^2+x-870=0$. **2.** $8+6x=5x^2$. **3.** $\frac{1}{2}x^2=14-3x^2$.

4. $x^2+7=4x$. **5.** $2x=x^2+5$. **6.** $(x+2)^2=4x+15$.

Form the equations whose roots are

7. $5, \ -3$. **8.** $-9, \ -11$. **9.** $a+b, \ a-b$.

10. $\frac{3}{2}, \ \frac{5}{6}$. **11.** $\frac{2}{3}a, \ -\frac{4}{5}a$. **12.** $0, \ \frac{7}{8}$.

13. If the equation $x^2+2(1+k)x+k^2=0$ has equal roots, what is the value of k?

14. Prove that the equation

$$3mx^2-(2m+3n)x+2n=0$$

has rational roots.

15. Without solving the equation $3x^2-4x-1=0$, find the sum, the difference, and the sum of the squares of the roots.

16. Shew that the roots of $a(x^2-1)=(b-c)x$ are always real.

Form the equations whose roots are

17. $3+\sqrt{5}, \ 3-\sqrt{5}$. **18.** $-2+\sqrt{3}, \ -2-\sqrt{3}$. **19.** $-\frac{a}{5}, \ \frac{b}{6}$.

20. $\frac{1}{2}(4\pm\sqrt{7})$. **21.** $\frac{a+b}{a-b}, \ \frac{a-b}{a+b}$. **22.** $\frac{a}{2b}, \ \frac{b}{2a}$.

If α, β are the roots of the equation $px^2+qx+r=0$, find the values of

23. $\alpha^2+\beta^2$. **24.** $(\alpha-\beta)^2$. **25.** $\alpha^2\beta+\alpha\beta^2$.

26. $\alpha^4+\beta^4$. **27.** $\alpha^5\beta^2+\alpha^2\beta^5$. **28.** $\frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha}$.

29. If α, β are the roots of $x^2-px+q=0$, and α^3, β^3 the roots of $x^2-Px+Q=0$, find P and Q in terms of p and q.

30. If α, β are the roots of $x^2-ax+b=0$, find the equation whose roots are $\frac{\alpha}{\beta^2}, \ \frac{\beta}{\alpha^2}$.

31. Find the condition that one root of the equation

$$ax^2+bx+c=0$$

may be double the other.

32. Form an equation whose roots shall be the cubes of the roots of the equation $2x(x-a)=a^2$.

33. Prove that the roots of the equation

$$(a+b)x^2-(a+b+c)x+\frac{c}{2}=0$$

are always real.

34. Shew that $(a+b+c)x^2-2(a+b)x+(a+b-c)=0$ has rational roots.

35. In the equation $px^2+qx+r=0$ the roots are in the ratio of l to m, prove that

$$(l^2+m^2)pr+lm(2pr-q^2)=0.$$

36. If one root of $x^2-9\cdot626x+13\cdot672408$ is $7\cdot894$, find the other root.

37. If one root of $x^2-2ax+a^2-4b^2-9c^2+12bc=0$ is $a+2b-3c$, what is the other root?

38. If α and β are the roots of the equation $3x^2-7x+2=0$, form the equation whose roots are:

(i) $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$; (ii) $\alpha+\beta$ and $\frac{1}{\alpha}+\frac{1}{\beta}$; (iii) $\frac{\alpha+\beta}{\alpha}$ and $\frac{\alpha+\beta}{\beta}$;

(iv) $\alpha\beta$ and $\frac{1}{\alpha\beta}$; (v) $\alpha+\frac{1}{\beta}$ and $\beta+\frac{1}{\alpha}$.

39. If the following equations have equal roots, find the values of k and the roots

(i) $x^2-(k+4)x+(k-2)^2=0$;

(ii) $(k^2-2)x^2+6kx+3(k^2+2)=0$;

(iii) $4x^2+5x+k^2=0$.

40. Form the equation whose roots are (i) greater by 2, (ii) less by 2, than those of $x^2-8x+12=0$.

41. Form the quadratic equation whose roots are $5\pm\sqrt{6}$.

If the roots of $x^2-px+q=0$ are two consecutive integers, prove that $p^2-4q-1=0$.

42. Form the equation whose roots are greater by 3 than those of $x^2+5x-8=0$.

43. Form the equation whose roots are greater by l than those of $ax^2+bx+c=0$.

44. If the roots of $ax^2+2bx+c=0$ are in the ratio of 2 to 3, prove $24b^2=25ac$.

45. What is the relation between p and q if one root of the equation $x^2+px+q=0$ is (i) 3 times, (ii) 4 times, (iii) m times, the other?

46. If one root of $ax^3+bx+c=0$ is the square of the other, shew that $b^3+ac^2+a^2c=3abc$.

47. If α and β are the roots of $x^2-px+q=0$, form the equations whose roots are

(i) $\dfrac{1}{\alpha}$ and $\dfrac{1}{\beta}$; (ii) $\alpha+\beta$ and $\alpha\beta$;

(iii) $\dfrac{1}{\alpha^2}$ and $\dfrac{1}{\beta^2}$; (iv) $\dfrac{\alpha}{\beta}$ and $\dfrac{\beta}{\alpha}$;

(v) $\dfrac{\alpha^2}{\beta}$ and $\dfrac{\beta^2}{\alpha}$; (vi) $\alpha+m$ and $\beta+m$;

(vii) $\alpha+\beta$ and $\dfrac{\alpha\beta}{\alpha+\beta}$; (viii) $(\alpha+\beta)^2$ and $(\alpha-\beta)^2$.

48. Find the sum of the squares of the roots of the equations,

(i) $x^2+6x+3=0$; (ii) $x^2+3ax+a^2=0$.

49. If $x^2-2x-a=0$ and $x^2-4x-5=0$ have a common root, shew that

$$a^2-18a+45=0.$$

50. If $2x^2-7x+6=0$ and $6x^2-kx+6=0$ have a common root, find the value of k, and the common root.

51. If the sum of the roots of $5kx^2-4x+k=0$ is equal to their product, find the value of k.

52. If α and β are the roots of $a^2cx^2+bcx+1=0$ and m and n are the roots of $ac^2x^2+abx+1=0$, shew that $\dfrac{\alpha+\beta}{m+n}=\left(\dfrac{\alpha\beta}{mn}\right)^2$.

53. Form the equation whose roots are the reciprocals of those of

$$ax^2+bx+c=0.$$

CHAPTER XXXIII.

MISCELLANEOUS EQUATIONS.

290. MANY kinds of miscellaneous equations may be solved by the ordinary rules for quadratic equations as explained in Art. 202; but others require some special artifice for their solution. These will be illustrated in the present chapter.

Example 1. Solve $\dfrac{x^2-6}{x}+\dfrac{5x}{x^2-6}=6.$

Write y for $\dfrac{x^2-6}{x}$; thus

$$y+\frac{5}{y}=6, \text{ or } y^2-6y+5=0;$$

whence $\qquad y=5, \text{ or } 1.$

$$\therefore \quad \frac{x^2-6}{x}=5, \text{ or } \frac{x^2-6}{x}=1;$$

that is, $\qquad x^2-5x-6=0, \text{ or } x^2-x-6=0.$

Thus $\qquad x=6, \ -1; \text{ or } x=3, \ -2.$

Example 2. Solve $3^{2x+3}-55=28(3^x-2).$

This equation may be written $3^3 \cdot 3^{2x}-28 \cdot 3^x+1=0.$

By writing y for 3^x, we obtain

$$27y^2-28y+1=0; \text{ that is, } (27y-1)(y-1)=0;$$

whence $\qquad y=\dfrac{1}{27}, \text{ or } 1.$

Thus $\qquad 3^x=\dfrac{1}{27}=3^{-3}, \text{ or } 3^x=1=3^0.$

and therefore $\qquad x=-3, \text{ or } 0.$

Example 3. Solve $2x^2-3\sqrt{2x^2-7x+7}=7x-3.$

On transposition, $(2x^2-7x)-3\sqrt{2x^2-7x+7}=-3.$

By putting $\sqrt{2x^2-7x+7}=y$, so that $2x^2-7x+7=y^2$, we obtain

$$(y^2-7)-3y=-3, \text{ or } y^2-3y-4=0;$$

whence $$y=4, \text{ or } -1.$$

Thus $$\sqrt{2x^2-7x+7}=4, \text{ or } \sqrt{2x^2-7x+7}=-1;$$

that is, $$2x^2-7x-9=0. \text{ or } 2x^2-7x+6=0.$$

From the first of these quadratics we obtain $x=\frac{9}{2}$, or -1, and from the second $x=2$, or $\frac{3}{2}$.

It should be noticed that in this solution we have tacitly assumed y to be the *positive* value of the expression $\sqrt{2x^2-7x+7}$, so that the roots obtained from the solution of $\sqrt{2x^2-7x+7}=-1$ will only satisfy the original equation in the modified form obtained by changing the sign of the radical.

Thus $x=\frac{9}{2}$, or -1 satisfies $2x^2-3\sqrt{2x^2-7x+7}=7x-3.$

and $x=2$, or $\frac{3}{2}$ satisfies $2x^2+3\sqrt{2x^2-7x+7}=7x-3.$

EXAMPLES XXXIII. a.

Solve the following equations :

1. $x^2+x+1=\frac{42}{x^2+x}.$

2. $\frac{x}{x^2-1}+\frac{x^2-1}{x}=2\frac{1}{6}.$

3. $\left(x+\frac{1}{x}\right)^2-4\left(x+\frac{1}{x}\right)=5.$

4. $8x^6+65x^3+8=0.$

5. $\frac{x+8}{x+12}+\frac{5}{x+4}=\frac{3x+14}{3x+8}.$

6. $4^x+8=9\,.\,2^x.$

7. $\frac{3x-6}{5-x}+\frac{11-2x}{10-4x}=3\frac{1}{2}.$

8. $3\sqrt{x}-3x^{-\frac{1}{2}}=8.$

9. $\left(x-\frac{6}{x}\right)^2+4x-\frac{24}{x}=5.$

10. $27x^{\frac{3}{2}}-1=26x^{\frac{3}{4}}.$

11. $7\sqrt{x-8}-\sqrt{21x+12}=2\sqrt{3}.$

12. $4^{2x+1}+16=65\,.\,4^x.$

13. $x+2=\sqrt{4+x\sqrt{8-x}}.$

14. $3^{\frac{x}{2}}+3^{-\frac{x}{2}}=2.$

15. $2x^2 - 2x + 2\sqrt{2x^2 - 7x + 6} = 5x - 6.$

16. $x^2 + 6\sqrt{x^2 - 2x + 5} = 11 + 2x.$

17. $2\sqrt{x^2 - 6x + 2} + 4x + 1 = x^2 - 2x.$

18. $\sqrt{4x^2 + 2x + 7} = 12x^2 + 6x - 119.$

19. $3x(3 - x) = 11 - 4\sqrt{x^2 - 3x + 5}.$

20. $x^2 - x + 3\sqrt{2x^2 - 3x + 2} = \frac{x}{2} + 7.$

21. $\sqrt{\frac{2 - x}{3x}} - \sqrt{\frac{3x}{2 - x}} = \frac{3}{2}.$

22. $\sqrt{\frac{a}{x}} - \sqrt{\frac{x}{a}} = \frac{a^2 - 1}{a}.$

23. $(a - b)x^2 + (b - c)x + c - a = 0.$

24. $a(b - c)x^2 + b(c - a)x + c(a - b) = 0.$

25. $\sqrt{a - x} + \sqrt{b - x} = \sqrt{a + b - 2x}.$

26. $\frac{1}{a - x} + \frac{1}{b - x} = \frac{1}{a - c} + \frac{1}{b - c}.$

27. $\sqrt{x - p} + \sqrt{x - q} = \frac{p}{\sqrt{x - q}} + \frac{q}{\sqrt{x - p}}.$

28. $\sqrt{(x - 2)(x - 3)} + 5\sqrt{\frac{x - 2}{x - 3}} = \sqrt{x^2 + 6x + 8}.$

29. $\sqrt{x^2 + 4x - 4} + \sqrt{x^2 + 4x - 10} = 6.$

30. $\sqrt[3]{x - a} - \sqrt[3]{x - b} = \sqrt[3]{b - a}.$

31. $\sqrt[3]{x + 17} - \sqrt[3]{x - 2} = 1.$

32. $\frac{\sqrt{x^2 + 5x} - \sqrt{2x + 1}}{\sqrt{x^2 + 5x} + \sqrt{2x + 1}} = \frac{1}{3}.$

33. $\sqrt[3]{x - 10} - \sqrt[3]{x - 12} = \sqrt[3]{2}.$

34. $(x^2 + 2x - 3)^2 - 7(x^2 + 2x) = 39.$

35. $\left(\frac{x}{x + 2}\right)^2 + \left(\frac{x + 2}{x}\right)^2 = \frac{82}{9}.$

291. No general methods can be given for the solution of simultaneous equations containing two or more unknowns. The simpler cases have been considered in Chapter XXVI. The following examples can be solved by special artifices:

EXAMPLES XXXIII. b.

1. $3x-2y=11,\quad 9x^2-4y^2=209.$

2. $x^3+y^3=91,\quad x^2y+xy^2=84.$

3. $x^3-y^3=335,\quad x^2y-xy^2=70.$

4. $x^2+xy+y^2=84,\quad x+\sqrt{xy}+y=14.$

5. $x^2+xy+y^2=189,\quad x-\sqrt{xy}+y=9.$

6. $\dfrac{3}{x^2}-\dfrac{1}{xy}-\dfrac{2}{y^2}=\dfrac{2}{9},\quad \dfrac{3}{x}+\dfrac{2}{y}=\dfrac{4}{3}.$

7. $\dfrac{2}{x^2}-\dfrac{3}{xy}-\dfrac{2}{y^2}=17,\quad \dfrac{1}{x}-\dfrac{2}{y}=1.$

8. $x^2y+y^2x=20,\quad \dfrac{1}{x}+\dfrac{1}{y}=\dfrac{5}{4}.$

9. $x^2-7xy+4y^2=34,\quad \dfrac{2x+y}{x-3y}-\dfrac{x-3y}{2x+y}=2\tfrac{2}{3}.$

10. $x^2-xy+x=35,\quad xy-y^2+y=15.$

11. $(x+y)^2+3(x-y)=30,\quad xy+3(x-y)=11.$

12. $(x-y)^2=3-2x-2y,\quad y(x-y+1)=x(y-x+1).$

13. $x^3+1=81(y^2+y),\quad x^2+x=9(y^3+1).$

14. $x^2-3xy=4y^2,\quad 2x^2-5xy-4y=8.$

15. $\sqrt{\dfrac{x}{y}}+\sqrt{\dfrac{y}{x}}=4\tfrac{1}{4},\quad \dfrac{x}{\sqrt{y}}+\dfrac{y}{\sqrt{x}}=16\tfrac{1}{4}.$

16. $\sqrt{\dfrac{x+y}{x-y}}+\sqrt{\dfrac{x-y}{x+y}}=\dfrac{34}{15},\quad \dfrac{x+y}{\sqrt{x-y}}+\dfrac{x-y}{\sqrt{x+y}}=\dfrac{152}{15}.$

17. $x+\dfrac{1}{y}=4,\quad y+\dfrac{1}{x}=1.$

CHAPTER XXXIV.

RATIO AND PROPORTION.

Ratio.

291. DEFINITION. **Ratio** is the relation which one quantity bears to another of the *same* kind, the comparison being made by considering what multiple, part, or parts, one quantity is of the other.

The ratio of A to B is usually written $A : B$. The quantities A and B are called the *terms* of the ratio. The first term is called the **antecedent**, the second term the **consequent.**

292. To find what multiple or part A is of B we divide A by B; hence the ratio $A : B$ may be measured by the fraction $\frac{A}{B}$, and we shall usually find it convenient to adopt this notation.

In order to compare two quantities they must be expressed in terms of the same unit. Thus the ratio of \$2 to 15 cents is measured by the fraction $\frac{2 \times 100}{15}$ or $\frac{40}{3}$.

Note. Since a ratio expresses the *number* of times that one quantity contains another, *every ratio is an abstract quantity.*

293. By Art. 151, $$\frac{a}{b} = \frac{ma}{mb};$$

and thus the ratio $a : b$ is equal to the ratio $ma : mb$; that is, *the value of a ratio remains unaltered if the antecedent and the consequent are multiplied or divided by the same quantity.*

294. Two or more ratios may be compared by reducing their equivalent fractions to a common denominator. Thus suppose $a : b$ and $x : y$ are two ratios. Now $\frac{a}{b} = \frac{ay}{by}$, and $\frac{x}{y} = \frac{bx}{by}$; hence the ratio $a : b$ is greater than, equal to, or less than the ratio $x : y$ according as ay is greater than, equal to, or less than bx.

295. The ratio of two fractions can be expressed as a ratio of two integers. Thus the ratio $\frac{a}{b} : \frac{c}{d}$ is measured by the fraction $\dfrac{\frac{a}{b}}{\frac{c}{d}}$, or $\frac{ad}{bc}$; and is therefore equivalent to the ratio $ad : bc$.

296. If either, or both, of the terms of a ratio be a surd quantity, then no two integers can be found which will *exactly* measure their ratio. Thus the ratio $\sqrt{2} : 1$ cannot be exactly expressed by any two integers.

297. Definition. If the ratio of any two quantities can be expressed exactly by the ratio of two integers the quantities are said to be **commensurable**; otherwise, they are said to be **incommensurable.**

Although we cannot find two integers which will exactly measure the ratio of two incommensurable quantities, we can always find two integers whose ratio differs from that required by as small a quantity as we please.

Thus $$\frac{\sqrt{5}}{4} = \frac{2{\cdot}236067\ldots}{4} = {\cdot}559016\ldots$$

and therefore $$\frac{\sqrt{5}}{4} > \frac{559016}{1000000} \text{ and } < \frac{559017}{1000000},$$

and it is evident that by carrying the decimals further, any degree of approximation may be arrived at.

298. Definition. Ratios are *compounded* by multiplying together the fractions which denote them; or by multiplying together the antecedents for a new antecedent, and the consequents for a new consequent.

Example. Find the ratio compounded of the three ratios

$$2a : 3b,\ 6ab : 5c^2,\ c : a.$$

The required ratio $= \frac{2a}{3b} \times \frac{6ab}{5c^2} \times \frac{c}{a} = \frac{4a}{5c}$.

299. When two or more ratios are equal, many useful propositions may be proved by introducing a single symbol to denote each of the equal ratios.

The proof of the following important theorem will illustrate the method of procedure.

If $$\frac{\mathbf{a}}{\mathbf{b}}=\frac{\mathbf{c}}{\mathbf{d}}=\frac{\mathbf{e}}{\mathbf{f}}=\ldots\ldots,$$

each of these ratios $$=\left(\frac{\mathrm{pa^n+qc^n+re^n+\ldots}}{\mathrm{pb^n+qd^n+rf^n+\ldots}}\right)^{\frac{1}{n}},$$

where p, q, r, n *are any quantities whatever.*

Let $$\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\ldots=k;$$

then $$a=bk,\ c=dk,\ e=fk,\ \ldots;$$

whence $$pa^n=pb^nk^n,\ qc^n=qd^nk^n,\ re^n=rf^nk^n,\ \ldots:$$

$$\therefore\ \frac{pa^n+qc^n+re^n+\ldots}{pb^n+qd^n+rf^n+\ldots}=\frac{pb^nk^n+qd^nk^n+rf^nk^n+\ldots}{pb^n+qd^n+rf^n+\ldots}$$
$$=k^n;$$

$$\therefore\ \left(\frac{pa^n+qc^n+re^n+\ldots}{pb^n+qd^n+rf^n+\ldots}\right)^{\frac{1}{n}}=k=\frac{a}{b}=\frac{c}{d}=\ldots.$$

By giving different values to p, q, r, n many particular cases of this general proposition may be deduced; or they may be proved independently by using the same method. For instance,

if $$\frac{a}{b}=\frac{c}{d}=\frac{e}{f},\ \text{each of these ratios}=\frac{a+c+e}{b+d+f};$$

a result which will frequently be found useful.

Example 1. If $\frac{x}{y}=\frac{3}{4}$ find the value of $\frac{5x-3y}{7x+2y}$,

$$\frac{5x-3y}{7x+2y}=\frac{\frac{5x}{y}-3}{\frac{7x}{y}+2}=\frac{\frac{15}{4}-3}{\frac{21}{4}+2}=\frac{3}{29}.$$

Example 2. Two numbers are in the ratio of 5 : 8. If 9 be added to each they are in the ratio of 8 : 11. Find the numbers.

Let the numbers be denoted by $5x$ and $8x$.

Then $$\frac{5x+9}{8x+9}=\frac{8}{11};$$

$$\therefore\ x=3.$$

Hence the numbers are 15 and 24.

EXAMPLES XXXIV. a.

1. If $x:y=5:7$, find the value of $x+y:y-x$.

2. If $\frac{x}{y}=3\frac{1}{3}$, find the value of $\frac{x-3y}{2x-5y}$.

3. If $b:a=2:5$, find the value of $2a-3b:3b-a$.

4. If $\frac{a}{b}=\frac{3}{4}$, and $\frac{x}{y}=\frac{5}{7}$, find the value of $\frac{3ax-by}{4by-7ax}$.

5. If $7x-4y:3x+y=5:13$, find the ratio $x:y$.

6. If $\frac{2a^2-3b^2}{a^2+b^2}=\frac{2}{41}$, find the ratio $a:b$.

7. Two numbers are in the ratio of $3:4$, and if 7 be subtracted from each the remainders are in the ratio of $2:3$. Find them.

8. What number must be taken from each term of the ratio $27:35$ that it may become $2:3$?

9. What number must be added to each term of the ratio $37:29$ that it may become $8:7$?

10. If $\frac{p}{b-c}=\frac{q}{c-a}=\frac{r}{a-b}$, shew that $p+q+r=0$.

11. If $\frac{x}{b+c}=\frac{y}{c+a}=\frac{z}{a-b}$, shew that $x-y+z=0$.

12. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, shew that the square root of

$$\frac{a^6b-2c^5e+3a^4c^3e^2}{b^7-2d^5f+3b^4cd^2e^2} \text{ is equal to } \frac{ace}{bdf}.$$

13. Prove that the ratio $la+mc+ne:lb+md+nf$ will be equal to each of the ratios $a:b$, $c:d$, $e:f$, if these be all equal; and that it will be intermediate in value between the greatest and least of these ratios if they be not all equal.

14. If $\frac{bx-ay}{cy-az}=\frac{cx-az}{by-ax}=\frac{z+y}{x+z}$, then will each of these fractions be equal to $\frac{x}{y}$, unless $b+c=0$.

15. If $\frac{2x-3y}{3z+y}=\frac{z-y}{z-x}=\frac{x+3z}{2y-3x}$, prove that each of these ratios is equal to $\frac{x}{y}$; hence shew that either $x=y$, or $z=x+y$.

Proportion.

300. Definition. When two ratios are equal, the four quantities composing them are said to be **proportionals.** Thus if $\frac{a}{b}=\frac{c}{d}$ then a, b, c, d are proportionals. This is expressed by saying that a is to b as c is to d, and the proportion is written

$$a:b::c:d;$$

or

$$a:b=c:d.$$

The terms a and d are called the *extremes*, b and c the *means*.

301. *If four quantities are in proportion, the product of the extremes is equal to the product of the means.*

Let a, b, c, d be the proportionals.

Then by definition $\frac{a}{b}=\frac{c}{d}$;

whence $ad=bc$.

Hence if any three terms of a proportion are given, the fourth may be found. Thus if a, c, d are given, then $b=\frac{ad}{c}$.

Conversely, if there are any four quantities, a, b, c, d, such that $ad=bc$, then a, b, c, d are proportionals; a and d being the extremes, b and c the means; or vice versâ.

302. Definition. Quantities are said to be in **continued proportion** when the first is to the second, as the second is to the third, as the third to the fourth; and so on. Thus a, b, c, d, ... are in continued proportion when

$$\frac{a}{b}=\frac{b}{c}=\frac{c}{d}=\ldots\ldots\ldots$$

If three quantities a, b, c are in continued proportion, then

$$a:b=b:c;$$

$$\therefore \quad ac=b^2. \qquad \text{[Art. 301.]}$$

In this case b is said to be a **mean proportional** between a and c; and c is said to be a **third proportional** to a and b.

303. If four quantities, a, b, c, d form a proportion, many other proportions may be deduced by the properties of fractions. The results of these operations are very useful.

(1) If $a : b = c : d$, then $b : a = d : c$.

For $\frac{a}{b} = \frac{c}{d}$; therefore $1 \div \frac{a}{b} = 1 \div \frac{c}{d}$;

that is, $$\frac{b}{a} = \frac{d}{c};$$

or $$b : a = d : c.$$

(2) If $a : b = c : d$, then $a : c = b : d$.

For $ad = bc$; therefore $\frac{ad}{cd} = \frac{bc}{cd}$;

that is, $$\frac{a}{c} = \frac{b}{d};$$

or $$a : c = b : d.$$

(3) If $a : b = c : d$, then $a + b : b = c + d : d$.

For $\frac{a}{b} = \frac{c}{d}$; therefore $\frac{a}{b} + 1 = \frac{c}{d} + 1$;

that is, $$\frac{a+b}{b} = \frac{c+d}{d};$$

or $$a + b : b = c + d : d.$$

(4) If $a : b = c : d$, then $a - b : b = c - d : d$.

For $\frac{a}{b} = \frac{c}{d}$; therefore $\frac{a}{b} - 1 = \frac{c}{d} - 1$;

that is, $$\frac{a-b}{b} = \frac{c-d}{d};$$

or $$a - b : b = c - d : d.$$

(5) If $a : b = c : d$, then $a + b : a - b = c + d : c - d$.

For by (3) $$\frac{a+b}{b} = \frac{c+d}{d};$$

and by (4) $$\frac{a-b}{b} = \frac{c-d}{d};$$

$\therefore$ by division, $$\frac{a+b}{a-b} = \frac{c+d}{c-d};$$

or $$a + b : a - b = c + d : c - d.$$

Several other proportions may be proved in a similar way.

Example 1. If $a : b = c : d = e : f$,

shew that $2a^2+3c^2-5e^2 : 2b^2+3d^2-5f^2 = ae : bf$.

Let $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=k$; then $a=bk$, $c=dk$, $e=fk$;

$$\therefore \frac{2a^2+3c^2-5e^2}{2b^2+3d^2-5f^2}=\frac{2b^2k^2+3d^2k^2-5f^2k^2}{2b^2+3d^2-5f^2}$$

$$=k^2=\frac{a}{b}\cdot\frac{e}{f}=\frac{ae}{bf},$$

or $2a^2+3c^2-5e^2 : 2b^2+3d^2-5f^2 = ae : bf$.

Example 2. Solve the equation $\frac{x^2+x-2}{x-2}=\frac{4x^2+5x-6}{5x-6}$.

$$\frac{x^2}{x-2}=\frac{4x^2}{5x-6};\qquad \text{[Art. 303 (4).]}$$

whence, dividing by x^2, which gives a solution $x=0$,

$$\frac{1}{x-2}=\frac{4}{5x-6};$$

whence $x=-2$,

and therefore the roots are $0, -2$.

Example 3. If

$(3a-6b+c+2d)(3a-6b-c+2d)=(3a-6b+c-2d)(3a+6b-c-2d)$,

prove that a, b, c, d are in proportion.

We have $\frac{3a+6b+c+2d}{3a-6b+c-2d}=\frac{3a+6b-c-2d}{3a-6b-c+2d}$, [Art. 301.]

and $\frac{2(3a+c)}{2(6b+2d)}=\frac{2(3a-c)}{2(6b-2d)}$, [Art. 303 (3), (4).]

and $\frac{3a+c}{3a-c}=\frac{6b+2d}{6b-2d}$. [Art. 303 (2).]

Again, $\frac{6a}{2c}=\frac{12b}{4d}$; [Art. 303 (3), (4).]

whence $a : b = c : d$.

EXAMPLES XXXIV. b.

Find a fourth proportional to

1. a, ab, c. **2.** $a^2, 2ab, 3b^2$. **3.** $x^3, xy, 5x^2y$.

Find a third proportional to

4. a^2b, ab. **5.** $x^3, 2x^2$. **6.** $3x, 6xy$. **7.** $1, x$.

Find a mean proportional between

8. a^2, b^2. **9.** $2x^3, 8x$. **10.** $12ax^2, 3a^3$. **11.** $27a^2b^3, 3b$.

If a, b, c are three proportionals, shew that

12. $a : a+b = a-b : a-c$.

13. $(b^2+bc+c^2)(ac-bc+c^2) = b^4+ac^3+c^4$.

If $a : b = c : d$, prove that

14. $ab+cd : ab-cd = a^2+c^2 : a^2-c^2$.

15. $a^2+ac+c^2 : a^2-ac+c^2 = b^2+bd+d^2 : b^2-bd+d^2$.

16. $a : b = \sqrt{3a^2+5c^2} : \sqrt{3b^2+5d^2}$.

Solve the equations :

17. $3x-1 : 6x-7 = 7x-10 : 9x+10$.

18. $x-12 : y+3 = 2x-19 : 5y-13 = 5 : 14$.

19. $\dfrac{x^2-2x+3}{2x-3} = \dfrac{x^2-3x+5}{3x-5}$. **20.** $\dfrac{2x-1}{x^2+2x-1} = \dfrac{x+4}{x^2+x+4}$.

21. If $$(a+b-3c-3d)(2a-2b-c+d)$$ $$=(2a+2b-c-d)(a-b-3c+3d)$$
prove that a b, c, d are proportionals.

22. If $a : b = c : d = e : f$, prove that

(i) $5a-7c+3e : 5b-7d+3f = c : d$.

(ii) $4a^2-5ace+6e^2f : 4b^2-5bde+6f^3 = ae : bf$.

(iii) $a^2ce : b^2df = 2a^4b^2+3a^2e^2-5e^4f : 2b^6+3b^2f^2-5f^5$.

23. If $a : b = x : y$, prove that

(i) $al+xm : bl+ym = ap+xq : bp+yq$.

(ii) $pa^2+qax+rx^2 : pb^2+qby+ry^2 = a^2+x^2 : b^2+y^2$.

MISCELLANEOUS EXAMPLES V.

1. Simplify $\dfrac{(a^2b^3)^{\frac{1}{6}}b^{-2}c^{\frac{1}{3}}}{a^{\frac{2}{3}}b^{-\frac{5}{4}}c^{\frac{1}{4}}}$, and find its value when $a=2$, $b=3$, and $c=432$.

2. If $\dfrac{a}{b} = \dfrac{c}{d}$, shew that

(1) $\dfrac{2a+3b}{3a-7b} = \dfrac{2c+3d}{3c-7d}$; (2) $\dfrac{a^2-c^2}{b^2-d^2} = \dfrac{(a+2c)(a+3c)}{(b+2d)(b+3d)}$.

3. If $\dfrac{x}{b-c}=\dfrac{y}{c-a}=\dfrac{z}{a-b}$, prove that

(1) $x+y+z=0$; (2) $(b+c)x+(c+a)y+(a+b)z=0$.

4. If $\dfrac{x}{1}=\dfrac{y}{2}=\dfrac{z}{3}$, prove that $\sqrt{5x^2+8y^2+7z^2}=5y$.

5. Simplify $\sqrt{45}+\sqrt{8}-\sqrt{80}+\sqrt{18}+\sqrt{7-\sqrt{40}}$.

6. If $\dfrac{a}{b}=\dfrac{c}{d}=\dfrac{e}{f}$, prove that each ratio is equal to

(1) $\sqrt[3]{\dfrac{4ac^2-3ce^3+2ace}{4bd^2-3cf^3+2bdf}}$; (2) $\sqrt[5]{\dfrac{6a^2c^2e-c^4ef+7ae^5}{6b^2d^2f-d^4f^2+7ad^5}}$.

---

7. If $3a+5b:3a-5b=3c+5d:3c-5d$,

prove that $a:b=c:d$.

8. Reduce to their simplest forms:

(1) $\dfrac{x^{a+b}}{x^{a-b}}+\dfrac{x^{a-b}}{x^{a+b}}$; (2) $\dfrac{(a+b)^{\frac{3}{2}}}{(a-b)^{\frac{1}{2}}}\times\sqrt{a^2-b^2}$.

9. When $x=-\dfrac{3a}{4}$, find the value of

$$\frac{x^2+ax+a^2}{x^3-a^3}-\frac{x^2-ax+a^2}{x^3+a^3}.$$

10. Simplify

(1) $\left\{\dfrac{a^{-1}b^3}{\sqrt[3]{a}}\right\}^{\frac{5}{4}}\div\sqrt[6]{\dfrac{a^2\sqrt{b}}{b^2}}$; (2) $\dfrac{2^{n+4}-2\times 2^n}{2^{n+2}\times 4}$.

11. Find the value of

$$\sqrt{19+4\sqrt{21}}+\sqrt{7}-\sqrt{12}-\sqrt{29-2\sqrt{28}}.$$

12. If $\dfrac{p}{bz-cy}=\dfrac{q}{cx-az}=\dfrac{r}{ay-bx}$, shew that

$ap+bq+cr=0$, and $xp+yq+zr=0$.

---

13. Simplify $$\frac{(a-b)^{\frac{1}{3}}\,.\,\sqrt[3]{a^2+2ab+b^2}}{\sqrt[3]{a^2-b^2}\times(a+b)^{-\frac{2}{3}}}.$$

14. If $a : b = c : d$, prove that

(1) $a+c : a+b+c+d = a : a+b$;

(2) $(a-b)-(c-d) = \dfrac{(a-b)(b-d)}{b}$.

15. Given that 4 is a root of the quadratic $x^2-5x+q=0$, find the value of q and the other root.

16. A person having 7 miles to walk increases his speed one mile an hour after the first mile, and finds that he is half an hour less on the road than he would have been had he not altered his rate. How long did he take?

17. If $(a+b+c)x = (-a+b+c)y = (a-b+c)z = (a+b-c)w$,

shew that $\qquad \dfrac{1}{y}+\dfrac{1}{z}+\dfrac{1}{w}=\dfrac{1}{x}$.

18. If α, β are the roots of $x^2+px+q=0$, shew that p, q are the roots of the equation

$$x^2+(\alpha+\beta-\alpha\beta)x-\alpha\beta(\alpha+\beta)=0.$$

---

19. Simplify $\dfrac{x^2-bc}{(a-b)(a-c)}+\dfrac{x^2-ca}{(b-c)(b-a)}+\dfrac{x^2-ab}{(c-a)(c-b)}$.

20. Solve the equations:

(1) $(x^2-5x+2)^2 = x^2-5x+22$.

(2) $\left(x^2+\dfrac{1}{x^2}\right)^2+4\left(x^2+\dfrac{1}{x^2}\right)=12$.

21. Prove that

$$(y-z)^3+(x-y)^3+3(x-y)(x-z)(y-z)=(x-z)^3.$$

22. Simplify $\dfrac{2\sqrt{10}}{3\sqrt{27}}\times\dfrac{15\sqrt{21}}{4\sqrt{15}}\div\dfrac{5\sqrt{14}}{7\sqrt{48}}$; and find the value of $\dfrac{1}{3\sqrt{5}-6}$, given that $\sqrt{5}=2{\cdot}236$.

23. Form the quadratic equation whose roots are

$$a+b+\sqrt{a^2+b^2} \text{ and } \frac{2ab}{a+b+\sqrt{a^2+b^2}}.$$

24. Solve the equations:

(1) $x^2(b-c)+ax(c-a)+a^2(a-b)=0$,

(2) $(x^2-px+p^2)(qx+pq+p^2)=qx^3+p^2q^2+p^4$.

CHAPTER XXXV.

EASY GRAPHS.

[*Arts.* 304–310 *may be read as soon as the student has had sufficient practice in substitutions involving negative quantities. Arts.* 311–324 *may be read after Simultaneous Equations of the first degree. Subsequent articles should be postponed until the student is acquainted with quadratic equations.*]

304. DEFINITION. Any expression which involves a variable quantity x, and whose value depends on that of x, is called a **function of x.**

Thus the expression $3x+8$ will have different values if different values are substituted for x, and is called a function of x of the first degree.

Similarly $2x^2+6x-7$, x^3-2x+1 are functions of x of the second and third degree respectively.

305. The words "function of x" are often briefly expressed by the symbol $f(x)$. If two quantities x and y are connected by a relation $y=f(x)$, by substituting a series of numerical values for x we can obtain a corresponding series of values for $f(x)$, that is for y.

Since in such a case the values of y *depend upon* the different values selected for x, it is sometimes convenient to call x the **independent variable**, and y the **dependent variable.**

306. Consider the function $x(9-x^2)$, and let its value be represented by y; so that $y=x(9-x^2)$.

Then, when $x=0$, $y=0\times 9=0$,

,, $x=1$, $y=1\times 8=8$,

,, $x=2$, $y=2\times 5=10$,

,, $x=3$, $y=3\times 0=0$,

,, $x=4$, $y=4\times(-7)=-28$,

and so on.

By proceeding in this way we can find as many values of the function as we please. But we are often not so much concerned with the actual values which a function assumes for different values of the variable as with *the way in which the value of the function changes.* These variations can be very conveniently represented by a **graphical** method which we shall now explain.

307. Two straight lines XOX′, YOY′ are taken intersecting at right angles in O, thus dividing the plane of the paper into four spaces XOY, YOX′, X′OY′, Y′OX, which are known as the first, second, third, and fourth quadrants respectively.

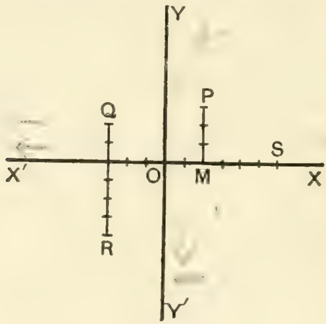


FIG. 1.

The lines X′OX, YOY′ are usually drawn horizontally and vertically; they are taken as lines of reference and are known as the **axis of x and y** respectively. The point O is called the **origin.** Values of x are measured from O along the axis of x, according to some convenient scale of measurement, and are called **abscissæ**, *positive* values being drawn to the *right* of O along OX, and *negative* values to the *left* of O along OX′.

Values of y are drawn parallel to the axis of y, from the ends of the corresponding abscissæ, and are called **ordinates.** These are *positive* when drawn *above* X′X, *negative* when drawn *below* X′X.

308. Suppose $y=3$, when $x=2$. To express this relation graphically we first mark off OM, 2 units in length, along OX; then at M we draw MP, 3 units in length, perpendicular to OX and above it. Thus the position of a point P is determined. Similarly any pair of corresponding values of x and y will determine a point relatively to the axes.

309. The abscissa and ordinate of a point taken together are known as its **coordinates.** A point whose coordinates are x and y is briefly spoken of as "the point (x, y)."

The process of marking the position of a point by means of its coordinates is known as **plotting the point.**

Example. *Plot the points*

(i) $(-3, 2)$; (ii) $(-3, -4)$; (iii) $(6, 0)$.

(i) We proceed as in Art. 5, but since x is negative we first take 3 units to the *left* of O, that is along OX′; then 2 units at right angles to OX′ and above it. The resulting point Q is in the second quadrant. See Fig. 1.

(ii) Here we may briefly describe the process as follows: Take 3 steps to the *left*, then 4 *down*; the resulting point R is in the third quadrant.

(iii) Take 6 steps to the *right*, then *no steps either up or down* from OX. Thus the resulting point S is on the axis of X.

Note. The coordinates of the origin are $(0, 0)$.

310. In practice it is convenient to use squared paper. Two intersecting lines should be chosen as axes, and slightly thickened to aid the eye, then one or more of the length-divisions may be taken as the linear unit.

We shall generally use paper ruled to tenths of an inch, but for greater clearness a larger scale will sometimes be adopted.

Example 1. *Plot the points* $(5, 2)$, $(-3, 2)$, $(-3, -4)$, $(5, -4)$ *on squared paper. Find the area of the figure determined by these points, assuming the divisions on the paper to be tenths of an inch.*

Taking the points in the order given, it is easily seen that they are represented by P, Q, R, S in Fig. 2, and that they form a rectangle which contains 48 squares. Each of these is *one-hundredth* part of a *square* inch. Thus the area of the rectangle is 0·48 of a square inch.

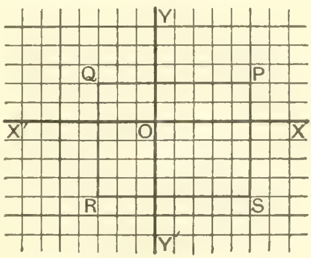


Fig. 2.

EXAMPLE 2. *The coordinates of the points* A *and* B *are* (7, 8) *and* (−5, 3) : *plot the points and find the distance between them.*

After plotting the points as in the diagram, we may find AB approximately by direct measurement.

Or we may proceed thus :

Draw through B a line par[l] to XX′ to meet the ordinate of A at C. Then ACB is a rt.-angled △ in which BC = 12, and AC = 5.

$$\text{Now } AB^2 = BC^2 + AC^2$$
$$= 12^2 + 5^2$$
$$= 144 + 25$$
$$= 169.$$
$$\therefore AB = 13.$$

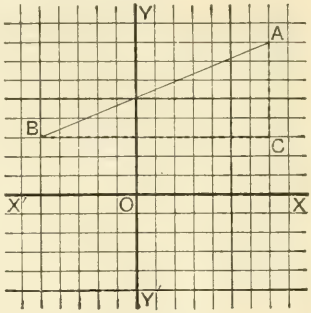


FIG. 3.

EXAMPLES XXXV. a.

[*The following examples are intended to be done mainly by actual measurement on squared paper; where possible, they should also be verified by calculation.*]

Plot the following pairs of points and draw the line which joins them :

1. (3, 0), (0, 6). **2.** (−2, 0), (0, −8).

3. (3, −8), (−2, 6). **4.** (5, 5), (−2, −2).

5. (−2, 6), (1, −3). **6.** (4, 5), (−1, 5).

7. Plot the points (3, 3), (−3, 3), (−3, −3), (3, −3), and find the number of squares contained by the figure determined by these points.

8. Plot the points (4, 0), (0, 4), (−4, 0), (0, −4), and find the number of units of area in the resulting figure.

9. Plot the points (0, 0), (0, 10), (5, 5), and find the number of units of area in the triangle.

10. Shew that the triangle whose vertices are (0, 0), (0, 6), (4, 3) contains 12 units of area. Shew also that the points (0, 0), (0, 6), (4, 8) determine a triangle of the same area.

11. Plot the points $(5, 6)$, $(-5, 6)$, $(5, -6)$, $(-5, -6)$. If each unit is supposed to represent one millimetre, find the area of the figure in square centimetres.

12. Plot the points $(1, 3)$, $(-3, -9)$, and shew that they lie on a line passing through the origin. Name the coordinates of other points on this line.

13. Plot the following points, and shew experimentally that each set lie in one straight line.

(i) $(9, 7)$, $(0, 0)$, $(-9, -7)$; (ii) $(-9, 7)$, $(0, 0)$, $(9, -7)$.

Explain these results theoretically.

14. Plot the following pairs of points; join the points in each case, and measure the coordinates of the mid-point of the joining line.

(i) $(4, 3)$, $(12, 7)$; (ii) $(5, 4)$, $(15, 16)$.

Shew *why* in each case the coordinates of the mid-point are respectively *half the sum of the abscissæ* and *half the sum of the ordinates* of the given points.

15. Plot the following pairs of points; and find the coordinates of the mid-point of their joining lines.

(i) $(0, 0)$, $(8, 10)$; (ii) $(8, 0)$, $(0, 10)$;

(iii) $(0, 0)$, $(-8, -10)$; (iv) $(-8, 0)$, $(0, -10)$.

16. Plot the following points, and calculate their distances from the origin.

(i) $(15, 8)$; (ii) $(-15, -8)$; (iii) $(2{\cdot}4'', {\cdot}7'')$; (iv) $(-{\cdot}7'', 2{\cdot}4'')$.

Check your results by measurement.

17. Plot the following pairs of points, and in each case calculate the distance between them.

(i) $(4, 0)$, $(0, 3)$; (ii) $(9, 8)$, $(5, 5)$;

(iii) $(15, 0)$, $(0, 8)$; (iv) $(10, 4)$, $(-5, 12)$;

(v) $(20, 12)$, $(-15, 0)$; (vi) $(20, 9)$, $(-15, -3)$.

Verify your calculation by measurement.

18. Plot the eight points $(0, 5)$, $(3, 4)$, $(5, 0)$, $(4, -3)$, $(-5, 0)$, $(0, -5)$, $(-4, 3)$, $(-4, -3)$, and shew that they are all equidistant from the origin.

19. Plot the two following series of points:

(i) $(5, 0)$, $(5, 2)$, $(5, 5)$, $(5, -1)$, $(5, -4)$;

(ii) $(-4, 8)$, $(-1, 8)$, $(0, 8)$, $(3, 8)$, $(6, 8)$.

Shew that they lie on two lines respectively parallel to the axis of y, and the axis of x. Find the coordinates of the point in which they intersect.

20. Shew that the points $(-3, 2)$, $(3, 10)$, $(7, 2)$ are the angular points of an isosceles triangle. Calculate and measure the lengths of the equal sides.

21. Explain by a diagram why the distances between the following pairs of points are all equal.

(i) $(a, 0)$, $(0, b)$; (ii) $(b, 0)$, $(0, a)$; (iii) $(0, 0)$, (a, b).

22. Draw the straight lines joining

(i) $(a, 0)$ and $(0, a)$; (ii) $(0, 0)$ and (a, a);

and prove that these lines bisect each other at right angles.

23. Find the perimeter of the triangle whose vertices are the points $(7, 0)$, $(0, 24)$, $(-10, 0)$.

24. Draw the figure whose angular points are given by

$$(0, -3),\ (8, 3),\ (-4, 8),\ (-4, 3),\ (0, 0).$$

Find the lengths of its sides, taking the points in the above order.

25. Plot the points $(13, 0)$, $(0, -13)$, $(12, 5)$, $(-12, 5)$, $(-13, 0)$, $(-5, -12)$, $(5, -12)$. Find their locus, (i) by measurement, (ii) by calculation.

26. Plot the points $(2, 2)$, $(-3, -3)$, $(4, 4)$, $(-5, -5)$, shewing that they all lie on a certain line through the origin. Conversely, shew that for *every* point on this line the abscissa and ordinate are equal.

27. If $y = 2x + 10$, find the values of y when x has the values 0, 1, 3, -2, -5. Plot the five points determined by these values, and shew experimentally that they lie on a straight line. Where does the line meet the axes?

28. By giving different values to x find by trial a series of points whose coordinates satisfy the equation $2y = 5x$. Shew that they all lie on a straight line through the origin.

[*It will be convenient here to take two tenths of an inch as the unit.*]

Graph of a Function.

311. Let $f(x)$ represent a function of x, and let its value be denoted by y. If we give to x a series of numerical values we get a corresponding series of values for y. If these are set off as abscissæ and ordinates respectively, we plot a succession of points. If *all* such points were plotted we should arrive at a line, straight or curved, which is known as the **graph** of the *function* $f(x)$, or the **graph** of the *equation* $y=f(x)$. Thus the graph of the *function* $2x-5$ is the same as the graph of the *equation* $y=2x-5$.

The variation of the *function* for different values of the variable x is exhibited by the variation of the *ordinates* as we pass from point to point.

In practice a few points carefully plotted will usually enable us to draw the graph with sufficient accuracy.

312. The student who has worked intelligently through the preceding examples will have acquired for himself some useful preliminary notions which will be of service in the examples on simple graphs which we are about to give. In particular, before proceeding further he should satisfy himself with regard to the following statements:

(i) The coordinates of the origin are (0, 0).

(ii) For every point on the axis of x the value of y is 0.

Thus the graph of $y=0$ *is the axis of* x.

(iii) For every point on the axis of y the value of x is 0.

Thus the graph of $x=0$ *is the axis of* y.

(iv) The graph of all points which have the same abscissa is a line parallel to the axis of y.

Thus on page 298, Ex. 19, (i) gives a line parallel to the axis of y, and this line is the graph of $x=5$.

(v) The graph of all points which have the same ordinate is a line parallel to the axis of x.

Thus on page 298, Ex. 19, (ii) gives a line parallel to the axis of x, and this line is the graph of $y=8$.

(vi) The distance of any point $P(x, y)$ from the origin is given by $OP^2=x^2+y^2$. (See Ex. 18, p. 297.)

EXAMPLE 1. *Plot the graph of* $y=x$.

When $x=0$, $y=0$; *thus the origin is one point on the graph.*

Also, when $x=1, 2, 3, \ldots -1, -2, -3, \ldots,$

$y=1, 2, 3, \ldots -1, -2, -3, \ldots$.

Thus the graph passes through **O**, and represents a series of points each of which has its ordinate equal to its abscissa, and is clearly represented by the straight line **POP′** in Fig. 4.

EXAMPLE 2. *Plot the graph of* $y=x+3$.

Arrange the values of x and y as follows:

x	3	2	1	**0**	−1	−2	−3	...
y	6	5	4	**3**	2	1	**0**	...

By joining these points we obtain a line **MN** parallel to that in Example 1.

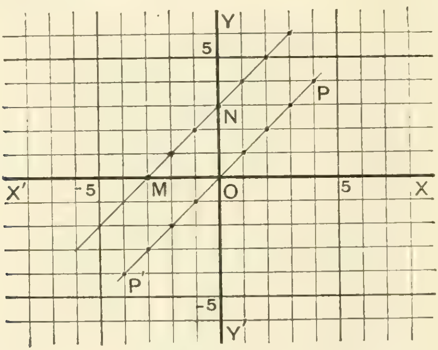


FIG. 4.

The results printed in larger and deeper type should be specially noted and compared with the graph. They shew that the distances **ON**, **OM** (usually called the *intercepts on the axes*) are obtained by separately putting $x=0$, $y=0$ in the equation of the graph.

Note. By observing that in Example 2 each ordinate is 3 units greater than the corresponding ordinate in Example 1, the graph of $y=x+3$ may be obtained from that of $y=x$ by simply producing each ordinate 3 units in the positive direction.

In like manner the equations

$$y=x+5, \quad y=x-5$$

represent two parallel lines on opposite sides of $y=x$ and equidistant from it, as the student may easily verify for himself.

EXAMPLE 3. *Plot the graphs represented by the equations:*

$$\text{(i) } 3y=2x\,; \quad \text{(ii) } 3y=2x+4\,; \quad \text{(iii) } 3y=2x-5.$$

First put the equations in the equivalent forms:

$$\text{(i) } y=\frac{2x}{3}\,; \quad \text{(ii) } y=\frac{2x}{3}+\frac{4}{3}\,; \quad \text{(iii) } y=\frac{2x}{3}-\frac{5}{3},$$

and in each case find values of y corresponding to

$$x=-3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3.$$

For example, in (i) we have the following values of y:

$$y=-2,\ -\tfrac{4}{3},\ -\tfrac{2}{3},\ 0,\ \tfrac{2}{3},\ \tfrac{4}{3},\ 2.$$

In plotting the corresponding points it will be found convenient to take *three* divisions of the paper as our unit.

The graphs will be found to be as in Fig. 5.

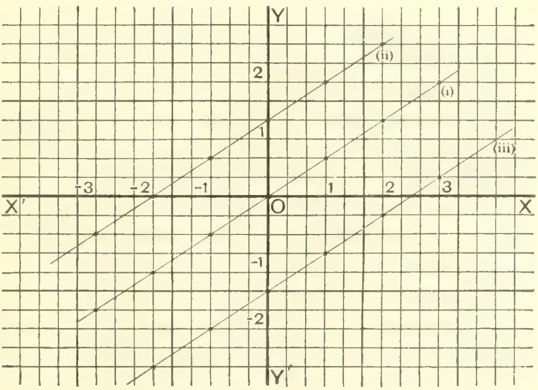


FIG. 5.

Each graph should be verified in detail by the student.

EXAMPLES XXXV. b.

[*Examples* 1–18 *are arranged in groups of three; each group should be represented on the same diagram so as to exhibit clearly the position of the three graphs relatively to each other.*]

Plot the graphs represented by the following equations :

1. $y=5x$. **2.** $y=5x-4$. **3.** $y=5x+6$.

4. $y=-3x$. **5.** $y=-3x+3$. **6.** $y=-3x-2$.

7. $y+x=0$. **8.** $y+x=8$. **9.** $y+4=x$.

10. $4x=3y$. **11.** $3y=4x+6$. **12.** $4y+3x=8$.

13. $x-5=0$. **14.** $y-6=0$. **15.** $5y=6x$.

16. $3x+4y=10$. **17.** $4x+y=9$. **18.** $5x-2y=8$.

19. Shew by careful drawing that the three last graphs have a common point whose coordinates are 2, 1.

20. Shew by careful drawing that the equations

$$x+y=10, \quad y=x-4$$

represent two straight lines at right angles.

21. Draw on the same axes the graphs of $x=5$, $x=9$, $y=3$, $y=11$. Find the number of units of area enclosed by these lines.

22. Taking one-tenth of an inch as the unit of length, find the area included between the graphs of $x=7$, $x=-3$, $y=-2$, $y-8$.

23. Find the area included by the graphs of

$$y=x+6, \quad y=x-6, \quad y=-x+6, \quad y=-x-6.$$

24. With one millimetre as linear unit, find in square centimetres the area of the figure enclosed by the graphs of

$$y=2x+8, \quad y=2x-8, \quad y=-2x+8, \quad y=-2x-8.$$

25. Draw the graphs of the following equations :

$$x+y=5, \quad 2x-y=10, \quad 2x+3y=-30, \quad 3y-x=15.$$

If the paper is ruled to tenths of an inch, shew that the graphs include an area of 1·5 sq. in.

313. The student should now be prepared for the following statements :

(i) For all numerical values of a the equation $y=ax$ represents a straight line through the origin.

If a is positive x and y have the same sign, and the line lies in the first and third quadrants ; if a is negative x and y have opposite signs, and the line lies in the second and fourth quadrant.

In either case a is called the **slope** of the line.

(ii) For all numerical values of a and b the equation $y=ax+b$ represents a line parallel to $y=ax$, and cutting off an intercept b from the axis of y.

The graph of $y=ax+b$ is fixed in position as long as a and b retain the same values.

If a alone is altered, the line will have a different direction but will still cut the axis of y at the same distance (b) from the origin.

If b alone is altered, the line will still be parallel to $y=ax$, but will cut the axis of y at a different distance from the origin, further or nearer according as b is greater or less.

Since the values a and b fix the position of the line we are considering in any one piece of work, they are called the **constants** of the equation.

Note. The *slope* of $y=ax+b$ is the same as that of $y=ax$

(iii) From the way in which the plotted points are determined from an equation, it follows that the graph passes through all points whose coordinates satisfy the equation, and through no other points.

314. Since every equation involving x and y only in the first degree can be reduced to one of the forms $y=ax$, $y=ax+b$, it follows that *every simple equation connecting two variables represents a straight line.* For this reason an expression of the form $ax+b$ is said to be a **linear function** of x, and an equation such as $y=ax+b$, or $ax+by+c=0$, is said to be a **linear equation.**

EXAMPLE. *Shew that the points* $(3, -4)$, $(9, 4)$; $(12, 8)$ *lie on a straight line, and find its equation.*

Assume $y=ax+b$ as the equation of the line. If it passes through the first two points given, their coordinates must satisfy this equation.

Substituting $x=3$, $y=-4$, we have

$$-4=3a+b. \quad \text{.................................(i)}$$

Again substituting $x=9$, $y=4$, we have

$$4=9a+b. \quad \text{.................................(ii)}$$

By solving equations (i) and (ii) we obtain

$$a=\frac{4}{3}, \; b=-8.$$

Hence $$y=\frac{4}{3}x-8, \text{ or } 4x-3y=24,$$

is the equation of the line passing through the first two points. Since $x=12$, $y=8$ satisfies this equation, the line also passes through $(12, 8)$. This example may be verified graphically by plotting the line which joins *any two* of the points and shewing that it passes through the third.

315. Since a straight line can always be drawn when *any* two points on it are known, in drawing a *linear* graph only two points need be plotted. The points where the line meets the axes can be readily found by putting $y=0$, $x=0$, successively in the equation, and these two points will always suffice, though they are not always the best to select.

EXAMPLE. *Draw the graph of* $4x-3y=13$.

If we find the intercepts on the axes we have

$$\text{when } y=0, \; x=\frac{13}{4} \text{ (intercept on the } x\text{-axis)},$$

$$\text{and when } x=0, \; y=-\frac{13}{3} \text{ (intercept on the } y\text{-axis)}.$$

As both of these values involve fractions of the unit, it would be difficult to draw the line with sufficient accuracy.

In such a case it is better to find by trial *integral* values of x and y which satisfy the equation.

Thus when $x=1$, $y=-3$, and when $y=1$, $x=4$.

The graph can now be drawn by joining the points $(1, -3)$, $(4, 1)$.

Application to Simultaneous Equations.

316. When there is only one simple equation connecting x and y, it is possible to find as many pairs of values of x and y as we please which satisfy the given equation. We now see that this is equivalent to saying that we may find as many points as we please on any given straight line. If, however, we have two *simultaneous* equations between x and y, there can only be one pair of values which will satisfy both equations. This is equivalent to saying that two straight lines can have only one common point.

Example. *Solve graphically the equations:*

$$\text{(i) } 3y - x = 6, \quad \text{(ii) } 3x + 5y = 38.$$

In (i) the intercepts on the axes are -6, 2. Thus the line is found by joining P$(-6, 0)$ and P$'(0, 2)$.

In (ii) when $x = 1$, $y = 7$, and when $y = 1$, $x = 11$.

Thus the line is found by joining Q$(1, 7)$ and Q$'(11, 1)$.

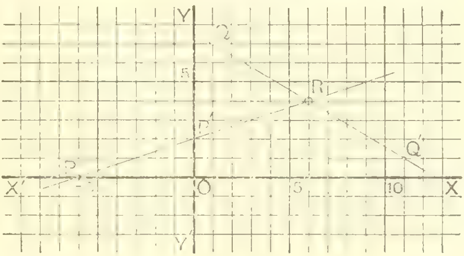


Fig. 6.

It is seen from the diagram that these lines intersect at the point R whose coordinates are 6, 4. Thus the solution of the given equations is $x = 6$, $y = 4$.

The student should verify this result by solving the equations algebraically by any of the methods applicable to simultaneous equations.

317. It will now be seen that the process of solving two linear simultaneous equations is equivalent to finding the coordinates of the point at which their graphs meet.

EXAMPLE. *Draw the graphs of*

$$\text{(i) } 5x+6y=60, \qquad \text{(ii) } 6y-x=24, \qquad \text{(iii) } 2x-y=7\,;$$

and shew that they represent three lines which meet in a point.

In (i) when $y=0$, $x=12$; when $x=0$, $y=10$.

Thus the intercepts on the axes are 12 and 10, and the graph is the line PP′.

In (ii) when $x=0$, $y=4$; when $x=12$, $y=6$, and the graph is the line joining Q (0, 4) to Q′ (12, 6).

In (iii) when $x=0$, $y=-7$; when $x=8$, $y=9$, and the graph is the line joining R (0, -7) to R′ (8, 9).

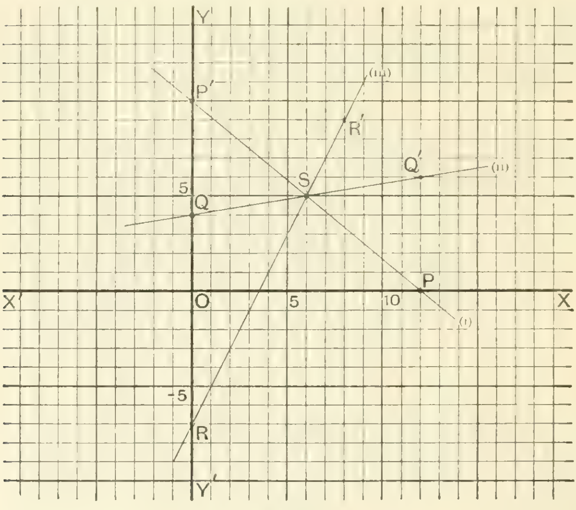


FIG. 7.

From the diagram it is evident that these three lines all pass through the point S whose coordinates are 6, 5.

318. Two simultaneous equations lead to no finite solution if they are inconsistent with each other. For example, the equations

$$x+3y=2, \quad 3x+9y=8$$

are inconsistent, for the second equation can be written $x+3y=2\frac{2}{3}$, which is clearly inconsistent with $x+3y=2$. The graphs of these two equations will be found to be two parallel straight lines which have no finite point of intersection.

Again, two simultaneous equations must be independent. The equations

$$4x+3y=1, \quad 16x+12y=4$$

are not independent, for the second can be deduced from the first by multiplying throughout by 4. Thus *any pair of values* which will satisfy one equation will satisfy the other. Graphically these two equations represent two coincident straight lines which of course have an unlimited number of common points.

EXAMPLES XXXV. c.

Solve the following equations graphically:

1. $y=2x+3$, $y+x=6$.

2. $y=3x+4$, $y=x+8$.

3. $y=4x$, $2x+y=18$.

4. $2x-y=8$, $4x+3y=6$.

5. $3x+2y=16$, $5x-3y=14$.

6. $6y-5x=18$, $4x=3y$.

7. $2x+y=0$, $y=\frac{4}{3}(x+5)$.

8. $2x-y=3$, $3x-5y=15$.

9. $2y=5x+15$, $3y-4x=12$.

10. Shew that the straight lines given by the equations

$$9y=5x+65, \quad 5x+2y+10=0, \quad x+3y=11,$$

meet in a point. Find its coordinates.

11. Prove by graphical representation that the three points (3, 0), (2, 7), (4, −7) lie on a straight line. Where does this line cut the axis of y?

12. Prove that the three points (1, 1), (−3, 4), (5, −2) lie on a straight line. Find its equation. Draw the graph of this equation, shewing that it passes through the given points.

13. Shew that the three points (3, 2), (8, 8), (−2, −4) lie on a straight line. Prove algebraically and graphically that it cuts the axis of x at a distance $1\frac{1}{3}$ from the origin.

319. Measurement on Different Scales. For the sake of simplicity we have hitherto measured abscissæ and ordinates on the same scale, but there is no necessity for so doing, and it will often be convenient to measure the variables on different scales suggested by the particular conditions of the question.

For example, in drawing the graph of $y = 11x + 6$,

when x has the values $\quad -2, -1, 0, 1, 2, 3,$

the corresponding values of y are $-16, -5, 6, 17, 28, 39.$

Thus some of the ordinates are much larger than the abscissæ, and rapidly increase as x increases.

On plotting these points with x and y measured on the same scale, it will be found that with a small unit the graph is inconveniently placed with regard to the axes. If a larger unit is employed the graph requires a diagram of inconvenient size.

[The student should prove this for himself experimentally.]

The inconvenience can be obviated by measuring the values of y on a considerably smaller scale than those of x.

For example, let us take $\frac{1}{20}$ of an inch as unit for y and one inch as unit for x; then the graph of $y = 11x + 6$ will be as in Fig. 8, in which the line has been drawn by joining the points (0, 6), (2, 28).

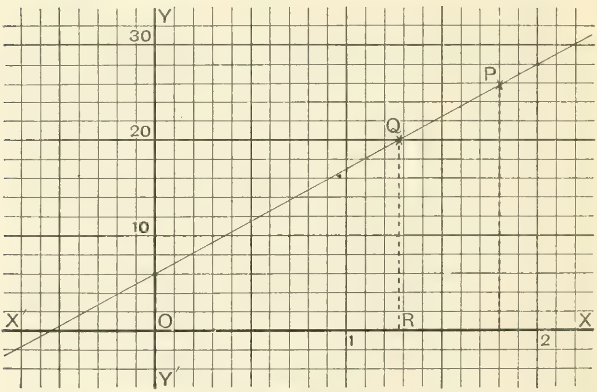


Fig. 8.

Speaking generally, whenever one variable increases much more rapidly than the other, a small unit should be chosen for the rapidly increasing variable and a large one for the other.

320. When a graph has been accurately drawn from plotted points, it can be used to *read off* (without calculation) corresponding values of the variables at intermediate points. Or if one coordinate of a point on the graph is known the other can be found by measurement. Sometimes, of course, results so obtained will only be approximate, but, as will be seen later, some of the most valuable results of graphical methods are arrived at in this way. The process is known as **interpolation.**

EXAMPLE. *From the graph of the expression* $11x+6$, *find its value when* $x=1 \cdot 8$. *Also find the value of* x *which will make the expression equal to* 20.

Put $y=11x+6$, then the graph is that given in Fig. 8. Now we see that $x=1 \cdot 8$ at the point P, and here $y=26$, nearly.

Again $y=20$ at the point Q; and $x=\text{OR}=1 \cdot 28$, approximately. In obtaining this last result we observe that OR is greater than $1 \cdot 2$ and less than $1 \cdot 3$, and we mentally divide the tenth in which R falls into *ten* equal parts (*i.e.* into *hundredths of the unit*) and judge as nearly as possible how many of these hundredths are to be added to $1 \cdot 2$.

EXAMPLES XXXV. d.

[*In some of the following Examples the scales are specified; in others the student is left to select suitable units for himself. When two or more equations are involved in the same piece of work, their graphs must all be drawn on the same scale. In every case the units employed should be marked on the axes.*]

1. By finding the intercepts on the axes draw the graphs of

(i) $15x+20y=6$; (ii) $12x+21y=14$.

In (i) take 1 inch for unit, and in (ii) take six tenths of an inch as unit. In each case explain why the unit is convenient.

2. Solve $y=10x+8$, $7x+y=25$ graphically.

[Unit for x, one inch; for y, one tenth of an inch.]

3. With the same units as in Ex. 2 draw the graph of the function $\dfrac{36-5x}{3}$. From the graph find the value of the function when $x=1 \cdot 8$; also find for what value of x the function becomes equal to 8.

4. On one diagram draw the graphs of

$$y=5x+11, \qquad 10x-2y=15.$$

What is the slope of these graphs? Find the length of the y-axis intercepted between them.

5. Draw the graphs of the equations:

$$3{\cdot}4x+5y=17, \quad x-y=0{\cdot}8, \quad y-0{\cdot}5x=0{\cdot}45;$$

and shew that they all pass through one point.

6. Draw the triangle whose sides are represented by the equations:

$$3y-x=9, \quad x+7y=11, \quad 3x+y=13;$$

and find the coordinates of the vertices.

7. With an inch as unit draw the triangle whose sides are given by the following equations, and find its vertices.

$$10y+2x=31, \quad y=3{\cdot}5x, \quad 5y-2x=6{\cdot}5.$$

8. I want a ready way of finding approximately 0·866 of any number up to 10. Justify the following construction. Join the origin to a point P whose coordinates are 10 and 8·66 (1 inch being taken as unit); then the ordinate of any point on OP is 0·866 of the corresponding abscissa. Read off from the diagram,

$$0{\cdot}866 \text{ of } 3, \; 0{\cdot}866 \text{ of } 6{\cdot}5, \; 0{\cdot}866 \text{ of } 4{\cdot}8, \text{ and } \frac{1}{0{\cdot}866} \text{ of } 5.$$

321. The last example gives a simple illustration of a graph used as a "ready reckoner." We shall now work two other examples of this kind.

Example 1. *Given that* 5·5 *kilograms are roughly equal to* 12·125 *pounds, shew graphically how to express any number of pounds in kilograms. Express* $7\frac{1}{2}$ *lbs. in kilograms, and* $4\frac{1}{4}$ *kilograms in pounds.*

Let y kilograms be equal to x pounds, then evidently we have $y=\frac{5{\cdot}5}{12{\cdot}125}x$, which is the equation of a straight line *through the origin.* Hence measuring pounds horizontally and kilograms vertically the required graph is obtained at once by joining the origin to the point whose coordinates are 12·125 and 5·5.

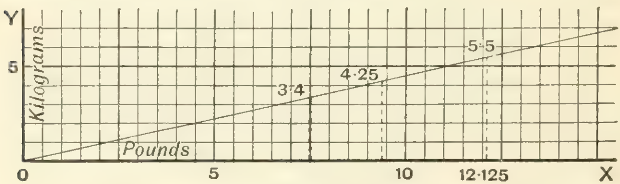


Fig. 9.

By measurement it will be found that $7\frac{1}{2}$ lbs. $=3{\cdot}4$ kilograms, and $4\frac{1}{4}$ kilograms $=9{\cdot}37$ lbs.

[The graph should be drawn by the student on a larger scale.]

EXAMPLE 2. *The expenses of a school are partly constant and partly proportional to the number of boys. The expenses were* \$650 *for* 105 *boys, and* \$742 *for* 128. *Draw a graph to represent the expenses for any number of boys; find the expenses for* 115 *boys, and the number of boys that can be maintained at a cost of* \$710.

If the total expenses for x boys are represented by \$$y$, the variable part may be denoted by \$$ax$, and the constant part by \$$b$. Hence x and y satisfy a linear equation $y = ax + b$, where a and b are constant quantities. Hence the graph is a straight line.

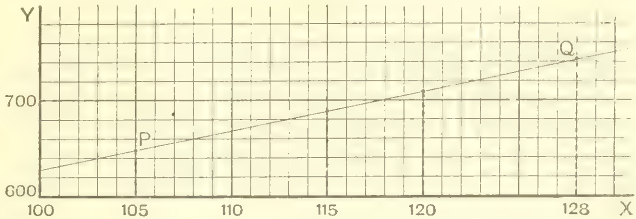


FIG. 10.

As the numbers are large, it will be convenient if we begin measuring ordinates at 600, and abscissæ at 100. This enables us to bring the requisite portion of the graph into a smaller compass. When $x = 105$, $y = 650$; and when $x = 128$, $y = 742$. Thus two points P and Q are found, and the line PQ is the required graph.

By measurement we find that when $x = 115$, $y = 690$; and that when $y = 710$, $x = 120$. Thus the required answers are \$690, and 120 boys.

EXAMPLES XXXV. d. *(Continued.)*

9. Given that 6·01 yards = 5·5 metres, draw the graph shewing the equivalent of any number of yards when expressed in metres.

Shew that 22·2 yards = 20·3 metres, approximately.

10. Draw a graph shewing the relation between equal weights in grains and grams, having given that 10·8 grains = 1·17 grams.

Express (i) 3·5 grams in grains.

(ii) 3·09 grains as a decimal of a gram.

11. If 3·26 in. are equivalent to 8·28 cm., shew how to find graphically the number of inches corresponding to a given number of centimetres. Obtain the number of inches in a metre, and the number of centimetres in a yard. Find the equation to the graph.

12. The highest marks gained in an examination were 136, and these are to be raised so that the maximum is 200. Shew how this may be done by means of a graph, and read off, to the nearest integer, the final marks of candidates who scored 61 and 49 respectively.

13. A man buys 100 eggs for \$1.20 and has to pay 30 cents for freight. He wishes to sell them so as to gain 20 per cent. on his whole outlay. Draw a graph to shew to the nearest cent the selling price of any number of eggs up to 100, and read off the price of 65. From the graph find the number of eggs which could be bought for \$2.25.

14. The highest and lowest marks gained in an examination are 297 and 132 respectively. These have to be reduced in such a way that the maximum for the paper (200) shall be given to the first candidate, and that there shall be a range of 150 marks between the first and last. Draw a graph from which the reduced marks may be read off, and find what marks should be given to candidates who gained 200, 262, 163 marks in the examination.

Find the equation between x, the actual marks gained, and y, the corresponding marks when reduced.

15. For a certain book it costs a publisher \$500 to prepare the type and 50*c*. to print each copy. Find an expression for the total cost in dollars of x copies. Make a diagram on a scale of 1 inch to 1000 copies, and 1 inch to \$500 to shew the total cost of any number of copies up to 5000. Read off the cost of 2500 copies, and the number of copies costing \$2750.

322. In all the cases at present considered the graph has been a straight line obtained by first selecting values of x and y which satisfy *an equation of the first degree*, and then drawing a line so as to pass through the plotted points. The method is quite general, and it is easy to see that it may be applied when the variables are connected by an equation *which is not linear*. In such a case it will be found that a line drawn through the plotted points will take the form of some *curve* differing in shape according to the equation which connects the variables. Before discussing such cases we may observe that, whenever two variable quantities depend on each other so that a change in one produces a corresponding change in the other, we can draw a graph to exhibit their variations without knowing any algebraical relation between them, *provided that we are furnished with a sufficient number of corresponding values accurately determined.*

But we frequently have to deal with cases in which a limited number of corresponding values of two variables have been obtained by observation or experiment. In such cases the data may involve inaccuracies, and consequently the position of the plotted points cannot be absolutely relied on. Moreover we cannot correct irregularities in the graph by plotting other points selected at discretion. One method of procedure is to join successive points by *straight* lines. The graph will then be represented by an irregular broken line, sometimes with abrupt changes of direction as we pass from point to point. In cases where no great accuracy of detail is required this simple method is often used to illustrate statistical results. A familiar instance is a Weather Chart giving the height of the barometer at equal intervals of time.

The chief disadvantage of this method is that, although it gives a general idea of the total change that has taken place between the plotted points, it furnishes no accurate information with regard to intermediate points.

EXAMPLE. *The readings of a thermometer taken at intervals of 2 hours beginning at* 10 a.m. *were* 62·5°, 64°, 69·6°, 69°, 66·5°, 65·7°.

Draw a chart to shew the changes of temperature.

Measuring degrees vertically and hours horizontally, with the scales indicated on the diagram, we obtain the broken line PQRSTV shewn in Fig. 11.

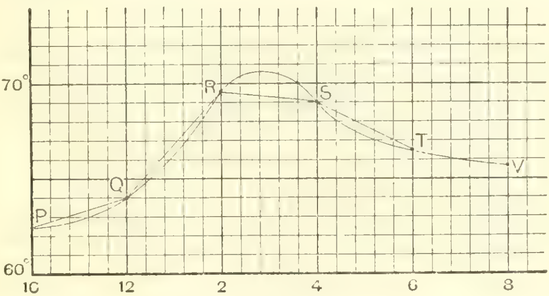


FIG. 11.

But it is contrary to experience to suppose that the abrupt changes of direction at Q and R accurately represent the change of temperature at noon and 2 p.m. respectively. Moreover, it is probable that the maximum temperature occurred at some time between 2 and 4, and not at the time represented at R, the highest of the

plotted points. Now if the chart had been obtained by means of a self-registering instrument, the graph (representing change from instant to instant instead of at long intervals) would probably have been somewhat like the continuous waving curve drawn through the points previously registered. From this it would appear that the maximum temperature occurred shortly before 3 p.m., and that TV (which represents a very gradual change) is the only portion of the broken line which records with any degree of accuracy the variation in temperature during two consecutive hours.

323. Although in the last example we were able to indicate the form of the curved line which from the nature of the case *seemed most probable*, it is evident that any number of curves can be drawn through a limited number of plotted points. In such a case the best plan is to draw a curve to lie as evenly as possible among the plotted points, passing through some perhaps, and with the rest fairly distributed on either side of the curve. As an aid to drawing an even continuous curve (usually called a *smooth* curve), a thin piece of wood or other flexible material may be bent into the requisite shape, and held in position while the line is drawn. A contrivance known as "Brooks' Flexible Curve" will often be found useful. When the plotted points lie approximately on a straight line, the simplest plan is to use a piece of tracing paper or celluloid on which a straight line has been drawn. When this has been placed in the right position the extremities can be marked on the squared paper, and by joining these points the approximate graph is obtained.

When the graph is linear it can be produced to any extent within the limits of the paper and so any value of one of the variables being determined, the corresponding value of the other can be read off. When large values are in question this method is inconvenient ; the following Example illustrates the method of procedure in such cases.

EXAMPLE. *Corresponding values of* x *and* y, *some of which are slightly inaccurate, are given in the following table* :

x	1	4	6·8	8	9·5	12	14·4
y	4	8	12·2	13	15·3	20	24·8

Draw the most probable graph and find its equation. Also find the value of y *corresponding to* $x = 80$.

Let 1 inch be taken to represent 5 units along OX, and 20 units along OY.

After carefully plotting the given points we see that a straight line can be drawn passing through three of them and lying evenly among the others. This is the required graph.

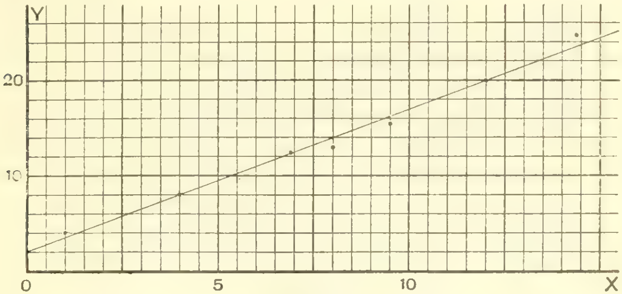


FIG. 12.

Assuming $y = ax + b$ for its equation, we can find the values of a and b by substituting the coordinates of two points through which the line passes.

Thus putting $x=4$, $y=8$, we obtain $8=4a+b$;
again, when $x=12$, $y=20$, we have $20=12a+b$.

By solving these equations we obtain $a=1·5$, $b=2$.

Hence the equation of the graph is $y=1·5x+2$, and the coordinates of any number of points on the line may now be found by trial.

Thus when $x=80$, $y=122$.

324. We shall now give an Example to illustrate a method common in the laboratory or workshop, the object being to determine the law connecting two variables when certain simultaneous values have been found by experiment or observation.

EXAMPLE. *In a certain machine P is the force in pounds required to raise a weight of W pounds. The following corresponding values of P and W were obtained experimentally:*

P	3·08 *	3·9	6·8	8·8	9·2	11 *	13·3
W	21	36·25	66·2	87·5	103·75	120	152·5

By plotting these values on squared paper draw the graph connecting P and W, and read off the value of P when $W=70$. Also determine a linear law connecting P and W; find the force necessary to raise a weight of 310 lbs., and also the weight which could be raised by a force of 180·6 lbs.

As the page is too small to exhibit the graphical work on a convenient scale we shall merely indicate the steps of the solution, which is similar in detail to that of the last example.

Plot the values of P vertically and the values of W horizontally. Taking 0·5 of an inch as unit for P, and 0·1 of an inch as unit for W, it will be found that a straight line can be drawn through the points corresponding to the results marked with an asterisk, and lying evenly among the other points. From this graph we find that when $W = 70$, $P = 7$.

Assume $P = aW + b$, and substitute for P and W from the values corresponding to the two points through which the line passes. By solving the resulting equations we obtain $a = 0{\cdot}08$, $b = 1{\cdot}4$. Thus the linear equation connecting P and W is $P = 0{\cdot}08\,W + 1{\cdot}4$.

This is called the **Law of the Machine.**

From this equation, when $W = 310$, $P = 26{\cdot}2$; and when $P = 180{\cdot}6$, $W = 2240$.

Thus a force of 26·2 lbs. will raise a weight of 310 lbs.; and when a force of 180·6 lbs. is applied the weight raised is 2240 lbs.

[The student should verify all the details of the work for himself.]

Note. The equation of the graph is not only useful for determining results difficult to obtain graphically, but it can always be used to check results found by measurement.

Example 2. *The following table gives statistics of the population of a certain country, where P is the number of millions at the beginning of each of the years specified.*

Year	1830	1835	1840	1850	1860	1865	1870	1880
P	20	22·1	23·5	29·0	34·2	38·2	41·0	49·4

Let t *be the time in years from* 1830. *Plot the values of P vertically and those of* t *horizontally and exhibit the relation between P and* t *by a simple curve passing fairly evenly among the plotted points. Find what the population was at the beginning of the years* 1848 *and* 1875.

Take 0·1 of an inch as unit in each case; also it will be convenient if we begin measuring abscissæ at 1830, and ordinates at 20.

The graph is given in Fig. 13 on the opposite page; it will be seen that it passes exactly through the extreme points and lies evenly among the others.

The populations in 1848 and 1875, at the points A and B respectively, will be found to be 27·8 millions and 45·3 millions.

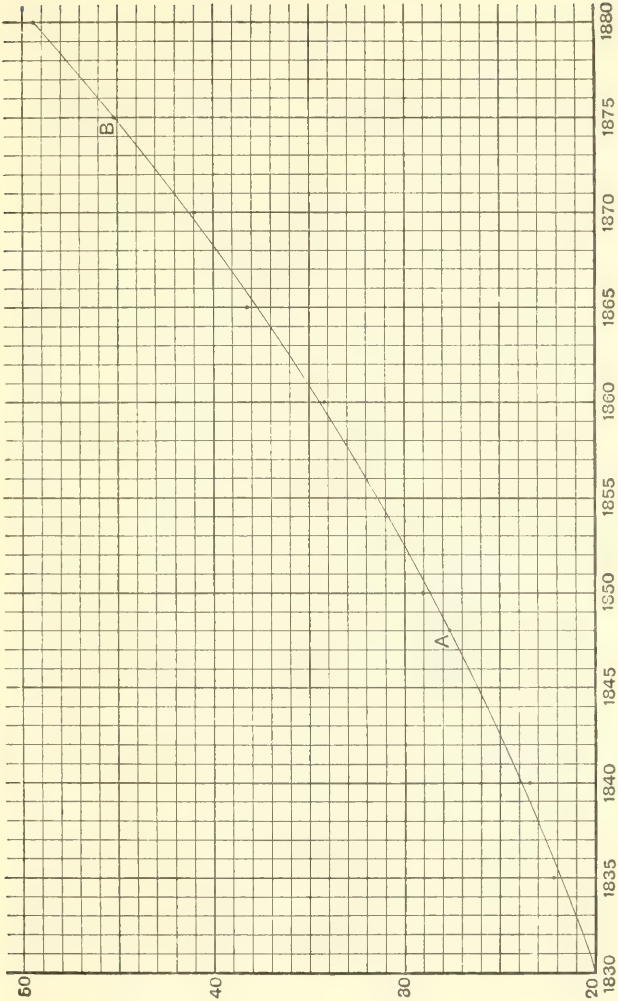

1880
1875
1870
1865
1860
1855
1850
1845
1840
1835
1830
60
40
30
20
A
B

Fig. 13.

EXAMPLES XXXV. e.

[*In Examples* 1–4 *the plotted points may be joined by straight lines. In other cases the graph is to be a straight line or smooth curve lying evenly among the plotted points.*]

1. In a term of 11 weeks a boy's places in his Form were as follows:

8, 6, 11, 10, 9, 6, 6, 4, 2, 1, 1.

Shew these results by means of a graph.

2. The mean heights of the barometer in inches for the first 10 days of January 1904, recorded at the Royal Observatory, were as follows:

29·21, 29·12, 29·00, 29·25, 29·37, 29·26, 29·46, 28·83, 28·66, 28·76.

Exhibit these variations by means of a chart.

3. The highest and lowest prices of Consols for the years 1895 to 1904 were as follows:

Year	'95	'96	'97	'98	'99	'00	'01	'02	'03	'04
Highest	$108\frac{1}{8}$	$113\frac{7}{8}$	$113\frac{7}{8}$	$113\frac{3}{8}$	$111\frac{1}{2}$	$103\frac{1}{4}$	$97\frac{7}{8}$	$97\frac{7}{8}$	$93\frac{5}{8}$	$91\frac{1}{4}$
Lowest	$103\frac{1}{2}$	$105\frac{1}{8}$	$110\frac{5}{8}$	$106\frac{3}{4}$	$97\frac{3}{4}$	$96\frac{3}{4}$	91	$92\frac{5}{16}$	$86\frac{7}{8}$	85

Make a chart to shew these variations graphically on the same diagram.

[A convenient scale will be: one inch to £10 vertically, beginning at 85, and 0·5 of an inch to 1 year horizontally.]

4. Make a chart to shew the variations in French Imports and Exports into Great Britain (in millions of pounds), for the years 1896 to 1903 inclusive, from the following data:

Imports	50·1	53·3	51·3	53·0	53·6	51·2	50·6	49·9
Exports	20·6	19·5	20·5	22·2	25·8	23·7	22·2	23·1

5. Corresponding values of x and y are given in the following table:

x	3	6·5	12	14	21	28·6	31·5
y	4	4·8	6·7	7	8·5	11	11·5

Draw the most probable graph, and find its equation. Find the value of x when $y=11{\cdot}5$, and the value of y when $x=10$.

6. Plot on squared paper the following measured values of x and y, and determine the most probable equation between x and y:

x	3	5	8·3	11	13	15·5	18·6	23	28
y	2	2·2	3·4	3·8	4	4·6	5·4	6·2	7·25

7. Corresponding values of x and y are given in the following table:

x	1	3·1	6	9·5	12·5	16	19	23
y	2	2·8	4·2	5·3	6·6	8·3	9	10·8

Supposing these values to involve errors of observation, draw the graph approximately, and determine the most probable equation between x and y. Find the correct value of y when $x=19$, and the correct value of x when $y=2{\cdot}8$.

8. At different ages the mean after-lifetime ("expectation of life") of males, calculated on the death rates of 1871–1880, was given by the following table:

Age	6	10	14	18	22	26	27
Expectation	50·38	47·60	44·26	40·96	37·89	34·96	34·24

Draw a graph to shew the expectation of any male between the ages of 6 and 27, and from it determine the expectation of persons aged 12 and 20.

9. The following table gives approximately the circumferences of circles corresponding to different radii:

C	15·7	20·1	31·4	44	52·2
r	2·5	3·2	5	7	8·3

Plot the values on squared paper, and from the graph determine the diameter of a circle whose circumference is 12·1 inches and the circumference of a circle whose radius is 2·8 inches.

10. For a given temperature, C degrees on a Centigrade are equal to F degrees on a Fahrenheit thermometer. The following table gives a series of corresponding values of F and C:

C	−10	−5	0	5	10	15	25	40
F	14	23	32	41	50	59	77	104

Draw a graph to shew the Fahrenheit reading corresponding to a given Centigrade temperature, and find the Fahrenheit readings corresponding to 12·5° C and 31° C.

By observing the form of the graph find the algebraical relation between F and C.

11. If W is the weight in ounces required to stretch an elastic string till its length is l inches, plot the following values of W and l:

W	2·5	3·75	6·25	7·5	10	11·25
l	8·5	8·7	9·1	9·3	9·7	9·9

From the graph determine the unstretched length of the string, and the weight the string will support when its length is 1 foot.

12. In a certain machine P is the force in pounds required to raise a weight of W pounds. The following corresponding values of P and W were obtained experimentally:

P	2·8	3·7	4·8	5·5	6·5	7·3	8	9·5	10·4	11·75
W	20	25	31·7	35·6	45	52·4	57·5	65	71	82·5

Draw the graph connecting P and W, and read off the value of P when $W=60$. Also determine the law of the machine, and find from it the weight which could be raised by a force of 31·7 lbs.

13. The connection between the areas of equilateral triangles and their bases (in corresponding square and linear units) is given by the following table:

Area	·43	1·73	3·90	6·93	10·82	15·59
Base	1	2	3	4	5	6

Illustrate these results graphically, and determine the area of an equilateral triangle on a base of 2·4 ft.

14. By measuring time along OX (1 inch for 1 hour), and distance along OY 1 (inch for 10 miles) shew that a line may be drawn from O through the points (1, 8) (2, 16), (3, 24), ... to indicate distance travelled towards Y in a specified time at 8 miles an hour.

A starts from Toronto at noon at 8 miles an hour; two hours later *B* starts, riding at 10 miles an hour. Find graphically at what time and at what distance from Toronto *B* overtakes *A*. At what times will *A* and *B* be 8 miles apart? If *C* rides after *B*, starting at 3 p.m. at 15 miles an hour, find from the graphs

(i) the distances between *A*, *B*, and *C* at 5 p.m.;

(ii) the time when *C* is 8 miles behind *B*.

15. With the same conditions as in Ex. 14, shew how to draw a line from Y to indicate distance travelled from Y towards O at 6 miles an hour.

If O and Y represent two towns 45 miles apart, and if *A* walks from Y to O at 6 miles an hour while *B* walks from O to Y at 4 miles an hour, both starting at noon, find graphically their time and place of meeting.

Also read off from the graphs

(i) the times when they are 15 miles apart;

(ii) *A*'s distance from Y at 6.15 p.m.

16. At 8 a.m. *A* starts from P to ride to Q which is 48 miles distant. At the same time *B* sets out from Q to meet *A*. If *A* rides at 8 miles an hour, and rests half an hour at the end of every hour, while *B* walks uniformly at 4 miles an hour, find graphically

(i) the time and place of meeting;

(ii) the distance between *A* and *B* at 11 a.m.;

(iii) at what time they are 14 miles apart.

325. We shall now give some graphs of functions of higher degree than the first.

Example. *Draw the graph of* $y = x^2$.

This is one of the most useful and interesting graphs the student will meet with; it is, therefore, important to plot the curve carefully on a suitable scale.

Take 0·4 of an inch as unit for x, and 0·1 of an inch for y, then *positive* values of x and y may be tabulated as follows:

x	0	0·5	1	1·5	2	2·5	3	3·5	4	...
y	0	0·25	1	2·25	4	6·25	9	12·25	16	...

Now if we take the following *negative* values of x

$$-0{\cdot}5,\ -1,\ -1{\cdot}5,\ -2,\ -2{\cdot}5,\ -3,\ -3{\cdot}5,\ -4,\ \ldots$$

we shall obtain the same series of values for y as before.

If the points we have now determined are plotted and connected by a continuous line drawn freehand, we shall obtain the curve shewn in Fig. 14.

There are three facts to be specially noted in this example.

(i) Since from the equation we have $x = \pm\sqrt{y}$, it follows that for every value of the ordinate we have two values of the abscissa, *equal in magnitude and opposite in sign.* Hence the graph is symmetrical with respect to the axis of y; so that after plotting with care enough points to determine the form of the graph in the first quadrant, its form in the second quadrant can be inferred without actually plotting any points in this quadrant. At the same time, in this and similar cases beginners are recommended to plot a few points in each quadrant through which the graph passes.

(ii) We observe that all the plotted points lie above the axis of x. This is evident from the equation; for since x^2 must be positive for all values of x, every ordinate obtained from the equation $y = x^2$ must be positive.

In like manner the student may shew that the graph of $y = -x^2$ is a curve similar in every respect to that in Fig. 14, but lying entirely below the axis of x.

(iii) As the numerical value of x increases that of y increases very rapidly. Hence, as there is no limit to the values which may be selected for x, it follows that the curve extends upwards and outwards to an infinite distance in both the first and second quadrants.

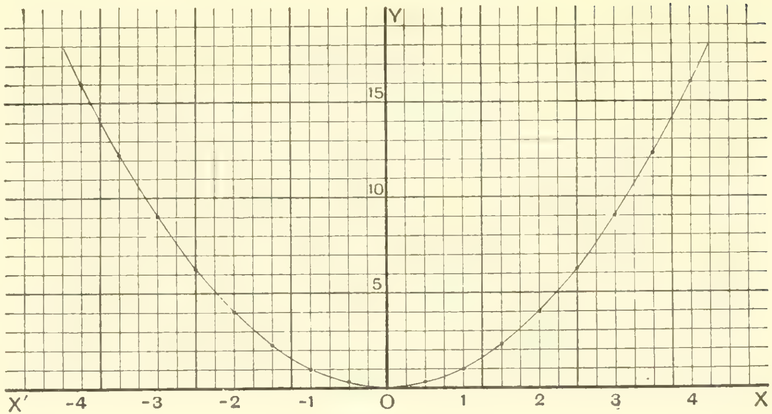


FIG. 14.

The student should draw this graph for himself with other units; for example, half an inch may be taken as unit on each axis. Or 1 inch may be taken as unit for x, and 0·1 of an inch as unit for y.

326. In any equation of the form $y = ax^2$, where a is constant, if a is a positive integer, the curve will be as in Fig. 14 but will rise more steeply in the direction of OY. If a is a positive fraction, we shall have a flatter curve, extending more rapidly to right and left of OY. If a is negative, the curve will lie below the y-axis, and will be steeper or flatter than the graph of $y = x^2$, according as a is greater or less than unity. In every case the axis of x is a tangent to the curve at the origin.

327. We shall now discuss the graphs of some quadratic functions of the form $ax^2 + bx + c$.

EXAMPLE. *Find the graph of* $y = 2x + \dfrac{x^2}{4}$.

Here the following arrangement will be found convenient:

x	3	2	1	0	−1	−2	−3	−4	−5	−6	−7	−8	−9
$2x$	6	4	2	0	−2	−4	−6	−8	−10	−12	−14	−16	−18
$\dfrac{x^2}{4}$	2·25	1	·25	0	·25	1	2·25	4	6·25	9	12·25	16	20·25
y	8·25	5	2·25	0	−1·75	−3	−3·75	−4	−3·75	−3	−1·75	0	2·25

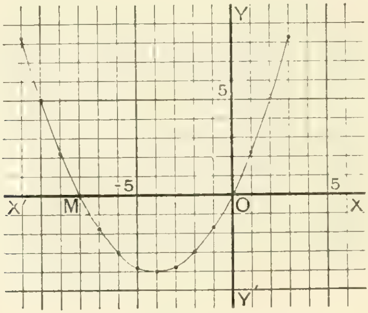


FIG. 15.

From the form of the equation it is evident that every positive value of x will yield a positive value of y, and that as x increases y

also increases. Hence the portion of the curve in the first quadrant lies as in Fig. 15, and can be extended indefinitely in this quadrant. In the present case only two or three positive values of x and y need be plotted, but more attention must be paid to the results arising out of negative values of x. It is found that the values of y are negative between $x=0$ and $x=-8$. When $x=-8$, $y=0$, and the curve crosses the x-axis; after this the values of y are positive.

328. In the last Example, since the value of $\frac{x^2}{4}+2x$ is represented by y, the *expression* $\frac{x^2}{4}+2x$ has a zero value when the ordinate is zero. Thus we can obtain the roots of the *equation* $\frac{x^2}{4}+2x=0$ by reading off the values of x at the points where the curve cuts the x-axis. These are $x=0$, $x=-8$, at the points O and M.

We can apply this method to an equation of any degree: thus if any function of x is represented by $f(x)$, a solution of the equation $f(x)=0$ may be obtained by plotting the graph of $y=f(x)$, and then measuring the intercepts made on the axis of x. These intercepts are values of x which make y equal to zero, and are therefore roots of $f(x)=0$.

329. In the graph of $y=x^2$ (Fig. 14) it will be noticed that as we pass from right to left along the curve the ordinate is constantly decreasing until it becomes zero at O; after this the ordinate begins to increase. The point at which this change takes place in a graph is known as a **turning point.** Thus the origin is a turning point of $y=x^2$, and of all curves represented by an equation of the form $y=ax^2$. Again in Fig. 15 there is a turning point at the point $(-4, -4)$. In each of these cases the algebraically least value of the ordinate is found at the turning point.

330. If a function gradually increases till it reaches a value a, which is algebraically greater than neighbouring values on either side, a is said to be a **maximum value** of the function.

If a function gradually decreases till it reaches a value b, which is algebraically less than neighbouring values on either side, b is said to be a **minimum** value of the function.

Let the function be represented by $f(x)$, then when $y=f(x)$ is treated graphically, it is evident that maximum and minimum values of $f(x)$ occur at the turning points, where the ordinates are algebraically greatest and least in the immediate vicinity of such points.

EXAMPLE. *Draw the graph of* $y = 3 - 4x - 4x^2$. *Thence find the roots of the equation* $4x^2 + 4x - 3 = 0$. *Shew that the expression* $3 - 4x - 4x^2$ *is positive for all real values of* x *between* 0·5 *and* −1·5, *and negative for all real values of* x *outside these limits. Also find the maximum value of* $3 - 4x - 4x^2$.

Take the unit for x four times as great as that for y, and use the following table of values:

x	2	1·5	1	0·5	0	−0·5	−1	−1·5	−2	−2·5
$-4x$	−8	−6	−4	−2	0	2	4	6	8	10
$-4x^2$	−16	−9	−4	−1	0	−1	−4	−9	−16	−25
y	−21	−12	−5	0	3	4	3	0	−5	−12

After plotting these points we have the graph given in Fig. 16.

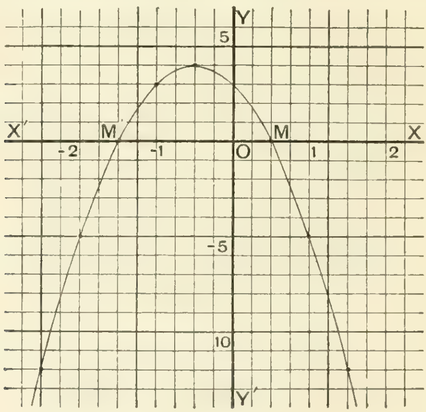


FIG. 16.

The roots of the equation $4x^2 + 4x - 3 = 0$ are the values of x which make y equal to 0. These are found at the points M and M′ where the curve cuts the x-axis. Thus the required roots are 0·5 and −1·5.

Again between the points M and M′ the graph lies above the x-axis; that is, the value of y, or $3 - 4x - 4x^2$, is positive so long as x lies between 0·5 and −1·5, and is negative for other values of x.

The maximum value of the expression $3-4x-4x^2$ is the value of the greatest ordinate in the graph, namely 4. This may also be obtained algebraically as follows :

$$3-4x-4x^2=3+1-(1+4x+4x^2)=4-(1+2x)^2.$$

Now $(1+2x)^2$ must be positive for all real values of x except $x=-\frac{1}{2}$, in which case it vanishes, and the value of the expression reduces to 4, which is the greatest value it can have.

Note. Another method of dealing with examples of this class will be found in Art. 334.

EXAMPLES XXXV. f.

1. Draw the graphs of (i) $y=x^2$, (ii) $y=8x^2$.

In (i) take 0·4″ as unit for x, 0·2″ as unit for y.

In (ii) ...... 1″ .... x, 0·1″ .... y.

2. On the same scale as in Ex. 1. (ii) draw the graph of $y=16x^2$. Shew that it may also be simply deduced from the graph of Ex. 1. (ii).

3. Plot the graph of $y=x^2$, taking 1 inch as unit on both axes, and using the following values of x.

$$-0·4,\ -0·3,\ -0·2,\ -0·1,\ 0,\ 0·1,\ 0·2,\ 0·3,\ 0·4.$$

4. Draw the graphs of $y=x^2$, and $x=y^2$, and shew that they have only one common chord. Find its equation.

5. From the graphs, and also by calculation, shew that $y=\frac{x^2}{8}$ cuts $x=-y^2$ in only two points, and find their coordinates.

6. Draw the graphs of

(i) $y^2=-4x$; (ii) $y=2x-\frac{x^2}{4}$; (iii) $y=\frac{x^2}{4}+x-2$.

In each case give the coordinates of the turning points.

7. Draw the graph of $y=x+x^2$. Shew also that it may be deduced from that of $y=x^2$.

8. Shew (i) graphically, (ii) algebraically, that the line $y=2x-3$ meets the curve $y=\frac{x^2}{4}+x-2$ in one point only. Find its coordinates.

9. Find graphically the roots of the following equations to 2 places of decimals :

(i) $\frac{x^2}{4}+x-2=0$; (ii) $x^2-2x=4$; (iii) $4x^2-16x+9=0$.

From the graphs deduce solutions of

(iv) $\frac{x^2}{4}+x-2=6$; (v) $x^2-2x=8$; (vi) $4x^2-16x+9=-6$.

10. On a large scale draw the graph of $x^2-7x+11$; hence find the roots of the equation $x^2-7x+11=0$, and the minimum value of the expression $x^2-7x+11$.

11. Find the minimum value of x^2-2x-4, and the maximum value of $5+4x-2x^2$.

12. Draw the graph of $y=(x-1)(x-2)$ and find the minimum value of $(x-1)(x-2)$. Measure, as accurately as you can, the values of x for which $(x-1)(x-2)$ is equal to 5 and 9 respectively. Verify algebraically.

13. Shew graphically that the expression x^2-2x-8 is negative for all values of x between -2 and 4, and positive of all values of x outside these limits.

331. The distance from the origin of any point P (x, y) is given by the relation $OP^2=x^2+y^2$. Hence any equation of the form $x^2+y^2=a^2$, where a is constant, represents a circle, of radius a, whose centre is at the origin, since every point (x, y) which satisfies the equation is at a constant distance a from the origin.

EXAMPLE. *Solve graphically the simultaneous equations*

(i) $x^2+y^2=41$, (ii) $y=2x-3$.

The graph of (i) is a circle. Since the equation is satisfied by $x=4$, $y=5$ (the point P), the graph may be drawn by describing a circle with centre O and radius OP.

The graph of (ii) is a straight line, which cuts the axes at the points (1·5, 0), (0, −3).

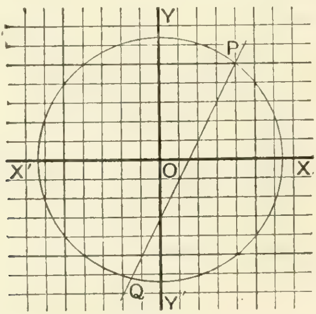


FIG. 17.

This line produced cuts the circle at P and Q. The coordinates of these points are (4, 5) and (−1·6, −6·2); thus the solution of the equations is given by

$$x=4,\ y=5,\ \text{and}\ x=-1{\cdot}6,\ y=-6{\cdot}2.$$

Infinite and Zero Values.

332. Consider the fraction $\frac{a}{x}$ in which the numerator a has a *certain fixed value*, and the denominator is a *quantity subject to change*; then it is clear that the smaller x becomes the larger does the value of the fraction $\frac{a}{x}$ become.

For instance

$$\frac{a}{\frac{1}{10}}=10a, \quad \frac{a}{\frac{1}{1000}}=1000a, \quad \frac{a}{\frac{1}{1000000}}=1000000a.$$

By making the denominator x sufficiently small the value of the fraction $\frac{a}{x}$ can be made as large as we please; that is, if x is made *less than any quantity that can be named*, the value of $\frac{a}{x}$ will become *greater than any quantity that can be named.*

A quantity less than any assignable quantity is called **zero** and is denoted by the symbol 0.

A quantity greater than any assignable quantity is called **infinity** and is denoted by the symbol ∞.

We may now say briefly

when $x=0$, *the value of* $\frac{a}{x}$ *is* ∞.

Again if x is a quantity which gradually increases and finally becomes *greater than any assignable quantity* the fraction becomes *smaller than any assignable quantity.* Or more briefly

when $x=\infty$, *the value of* $\frac{a}{x}$ *is* 0.

It should be observed that when the symbols for zero and infinity are used in the sense above explained, they are subject to the rules of signs which affect other algebraical symbols. Thus we shall find it convenient to use a concise statement such as "when $x=+0$, $y=+\infty$" to indicate that when a *very small and positive* value is given to x, the corresponding value of y is *very large and positive.*

EXAMPLE. *Find the graph of* xy = 4. *Shew that it consists of two infinite branches, one in the first and the other in the third quadrant.*

The equation may be written in the form $y = \frac{4}{x}$,

from which it appears that when $x = 0$, $y = \infty$ and when $x = \infty$, $y = 0$. Also y is positive when x is positive, and negative when x is negative. Hence the graph must lie entirely in the first and third quadrants.

Take the positive and negative values of the variables separately.

(1) *Positive values:*

x	0	1	2	3	4	5	6	...	∞
y	∞	4	2	$1\frac{1}{3}$	1	·8	$\frac{2}{3}$	...	0

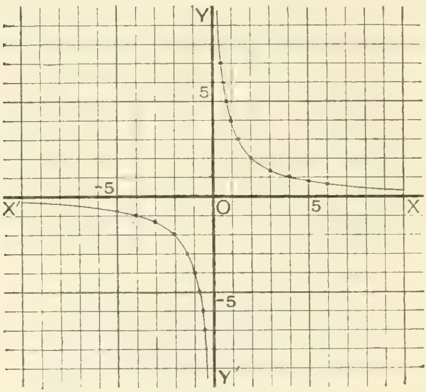


FIG. 18.

Graphically these values shew that as we recede further and further from the origin on the x-axis in the positive direction, the values of y are positive and become smaller and smaller. That is the graph is continually approaching the x-axis in such a way that by taking a sufficiently great positive value of x we obtain a point on the graph as near as we please to the x-axis but never actually reaching it until $x = \infty$. Similarly, as x becomes smaller and smaller the graph approaches more and more nearly to the positive end of the y-axis, never actually reaching it as long as x has any finite positive value, however small.

(2) *Negative values:*

x	-0	-1	-2	-3	-4	-5	...	$-\infty$
y	$-\infty$	-4	-2	$-1\frac{1}{3}$	-1	$-\cdot 8$	...	-0

The portion of the graph obtained from these values is in the third quadrant as shewn in Fig. 18, and exactly similar to the portion already traced in the first quadrant. It should be noticed that as x passes from $+0$ to -0 the value of y changes from $+\infty$ to $-\infty$. Thus the graph which in the first quadrant has run away to an infinite distance on the positive side of the y axis, reappears in the third quadrant coming from an infinite distance on the negative side of that axis. Similar remarks apply to the graph in its relation to the x-axis.

333. In the simpler cases of graphs, sufficient accuracy can usually be obtained by plotting a few points, and there is little difficulty in selecting points with suitable coordinates. But in other cases, and especially when the graph has infinite branches, more care is needed. The most important thing to observe are (1) the values for which the function $f(x)$ becomes zero or infinite: and (2) the values which the function assumes for zero and infinite values of x. In other words, we determine the *general character* of the curve in the neighbourhood of the origin, the axes, and infinity. Greater accuracy of detail can then be secured by plotting points at discretion. The selection of such points will usually be suggested by the earlier stages of our work.

The existence of symmetry about either of the axes should also be noted. When an equation contains no *odd* powers of x, the graph is symmetrical with regard to the axis of y. Similarly the absence of odd powers of y indicates symmetry about the axis of x. [Compare Example in Art. 325.]

EXAMPLE. *Solve the following pairs of equations graphically:*

$$\text{(i)}\ \left.\begin{aligned} x-y&=2 \\ xy&=35 \end{aligned}\right\};\qquad \text{(ii)}\ \left.\begin{aligned} x^2+y^2&=74 \\ xy&=35 \end{aligned}\right\}.$$

In each case we shall require the graph of $xy=35$. Proceeding as in the example of Art. 332, we find that the curve lies in the first and third quadrants.

In (i) $x-y=2$ is a straight line QS making intercepts 2 and -2 on the axes.

In (ii) $x^2+y^2=74$ is a circle. Since the equation is satisfied by $x=5$, $y=7$, the graph can be drawn by finding this point (P), and describing a circle with centre O and radius OP.

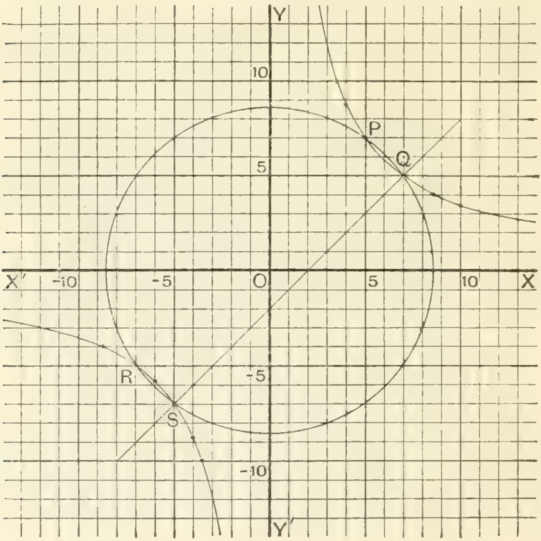


FIG. 19.

The roots of (i) are the coordinates of Q and S; that is,

$$x=7,\ y=5;\ \text{or}\ x=-5,\ y=-7.$$

The roots of (ii) are the coordinates of P, Q, R, and S; that is,

$$x=5,\ y=7;\ x=7,\ y=5;\quad x=-7,\ x=-5;\quad x=-5,\ y=-7.$$

334. Combination of two graphs. The method employed in the example in Art. 330 is quite general, and may be applied to functions of the third or higher degree, but the same results may often be more readily obtained by combining two graphs in the manner illustrated below.

EXAMPLE. *Solve the equation* $2x^2 - x - 3 = 0$ *graphically. Between what values of* x *is the expression* $2x^2 - x - 3$ *positive?*

Write the equation in the form $x^2 = \frac{x}{2} + \frac{3}{2}$.

Put $y_1 = x^2$..............(i), and $y_2 = \frac{x}{2} + \frac{3}{2}$..............(ii).

and plot the graphs of these equations, taking the x unit twice as great as the y unit.

For (i) we may use the values:

x	0	±0·5	±1	±1·5	±2
y	0	0·25	1	2·25	4

Thus we obtain the curve POQ.

The intercepts of (ii) on the axes are -3, 1·5; thus the graph of (ii) is the straight line PQ.

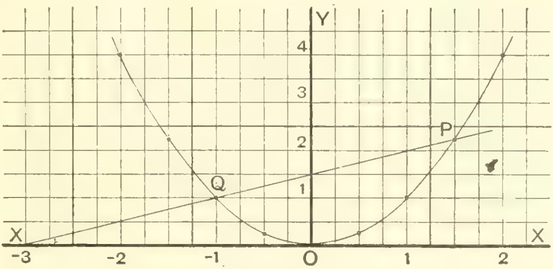


FIG. 20.

At the points of intersection, P and Q, the ordinates of (i) and (ii) are equal, that is $y_1 = y_2$; and the values of x at these points are 1·5 and -1. Hence for these values of x we have

$$x^2 = \frac{x}{2} + \frac{3}{2}, \text{ or } 2x^2 - x - 3 = 0.$$

Thus the roots of the equation $2x^2 - x - 3 = 0$ are furnished by the abscissæ of the common points of the graphs of $y = x^2$ and $y = \frac{x}{2} + \frac{3}{2}$.

Again, the expression $2x^2 - x - 3$ is positive or negative according as y_1 is greater or less than y_2. From the graph we see y_1 is less than y_2 between Q and P, that is between $x = -1$ and $1 \cdot 5$, and y_1 is greater than y_2 for all other values of x. Hence $2x^2 - x - 3$ is positive for all values of x except such as lie between -1 and $1 \cdot 5$.

335. The solution of the last example might have been effected equally well by drawing the graphs of $y = 2x^2$ and $y = x + 3$. But if a number of quadratic equations have to be solved graphically it is convenient to reduce them to the form $x^2 = px + q$ as a first step. The graph of $y = x^2$ can then be plotted once for all on a suitable scale, and the line $y = px + q$ can be readily drawn for different values of p and q.

EXAMPLES XXXV. g.

1. Solve the following equations graphically.

(i) $x^2 + y^2 = 53$, $y - x = 5$; (ii) $x^2 + y^2 = 100$, $x + y = 14$;

(iii) $x^2 + y^2 = 34$, $2x + y = 11$; (iv) $x^2 + y^2 = 36$, $4x + 3y = 12$.

[Approximate roots to be given to one place of decimals.]

2. Plot the graphs of $x^2 + y^2 = 25$, $3x + 4y = 25$, and examine their relation to each other where they meet. Verify the result algebraically.

3. By the method of Art. 334 find graphically the roots of the following equations to two places of decimals:

(i) $\dfrac{x^2}{4} + x - 2 = 0$; (ii) $x^2 - 2x = 4$; (iii) $4x^2 - 16x + 9 = 0$.

4. Solve graphically the equation $3 + 6x = x^2$, and find the maximum value of the expression $3 + 6x - x^2$.

5. Shew by the method of Art. 334 that the expression $4x^2 + 4x - 3$ is negative of all real values of x between $0 \cdot 5$ and $-1 \cdot 5$, and positive for all real values of x outside those limits.

6. Draw the graphs of x^2 and of $3x + 1$. By means of them find approximate values for the roots of $x^2 - 3x - 1 = 0$.

7. Shew graphically that the expression $x^2 - 4x + 7$ is positive for all real values of x.

8. On the same axes draw the graphs of

$$y=x^2, \quad y=x+6, \quad y=x-6, \quad y=-x+6, \quad y=-x-6.$$

Hence discuss the roots of the four equations

$$x^2-x-6=0, \quad x^2-x+6=0, \quad x^2+x-6=0, \quad x^2+x+6=0.$$

9. If x is real prove graphically that $5-4x-x^2$ is not greater than 9; and that $4x^2-4x+3$ is not less than 2. Between what values of x is the first expression positive?

10. The reciprocal of a number is multiplied by 2·25 and the product is added to the number. Find graphically what the number must be if the resulting expression has the least possible value.

11. Shew graphically that the expression $4x^2+2x-8{\cdot}75$ is positive for all real values of x except such as lie between 1·25 and $-1{\cdot}75$. For what value of x is the expression a minimum?

12. Solve the following pairs of equations graphically:

(i) $x+y=15$, $xy=36$; (ii) $x-y=3$, $xy=18$; (iii) $x^2+y^2=13$, $xy=6$.

Miscellaneous Applications of Graphs.

336. When two quantities x and y are so related that a change in one produces a proportional change in the other, their variations can always be expressed by an equation of the form $y=ax$, where a is constant. Hence in all such cases the graph which exhibits their variations is a straight line through the origin, and only one other point is required to determine the graph. For instance, such examples as deal with work and time, distance and time (when the speed is uniform), quantity and cost of material, principal and simple interest at a given rate per cent., may all be illustrated by linear graphs through the origin.

Note. It must be admitted that solutions of this kind are often very artificial, and they should be regarded mainly as exercises in ingenuity. The graphical treatment of a quite simple problem may often prove cumbrous and elaborate in detail, and in such a case a straightforward arithmetical or algebraical solution is to be preferred. For example, it is an unprofitable waste of time and skill to devise graphical solutions for certain types of easy problems which at most require only a few lines of very simple Arithmetic or Algebra. When, however, the answer to a question involves several allied results (as in Examples XXXV. c. 14-16) a graphical method is often useful and interesting.

EXAMPLE 1. *P and Q are two towns 30 miles apart. At 1 p.m. X starts to walk from Q to P at 3 mi. an hour, and after walking two hours finds it necessary to run back for his watch. This he does at $6\frac{2}{3}$ mi. an hour, and after a delay of 6 minutes he again starts from Q, at 4 mi. an hour. Meanwhile Y starting from P at 1 p.m. sets out for Q at 4 mi. an hour; after walking for two hours, he spends half an hour with a friend from whom he borrows a bicycle on which he continues his journey at 12 mi. an hour. Draw graphs to shew the position of each man relative to P and Q at any time between 1 p.m. and 5.30 p.m. Also from the graphs find*

(i) *when and where X and Y meet;*

(ii) *at what times respectively they were 18 mi. and 8 mi. apart.*

In Fig. 21, on the opposite page, time is measured horizontally (1 inch to 1 hour), and distance vertically (1 inch to 10 miles). Thus each division on the horizontal axis represents 6 minutes and each division on the vertical axis stands for 1 mile.

The graph shewing the course of X is drawn downwards from Q; similarly Y's course is shewn by a graph drawn upwards from P.

At 3 p.m. X has gone 6 miles, therefore if A is taken 0·6 inch below the point which marks 3 p.m., QA is his graph for the first 2 hours.

To get back to Q at $6\frac{2}{3}$ miles an hour will take $6 \div 6\frac{2}{3}$, or $\frac{9}{10}$ of an hour. Hence B is the next point on his graph.

The delay at Q before he starts again at 4 miles an hour at 4 p.m. is represented by BC, which denotes 6 minutes. If D is taken 0·4 inch, representing 4 miles, vertically below 5 p.m., the line CDE completes the graph.

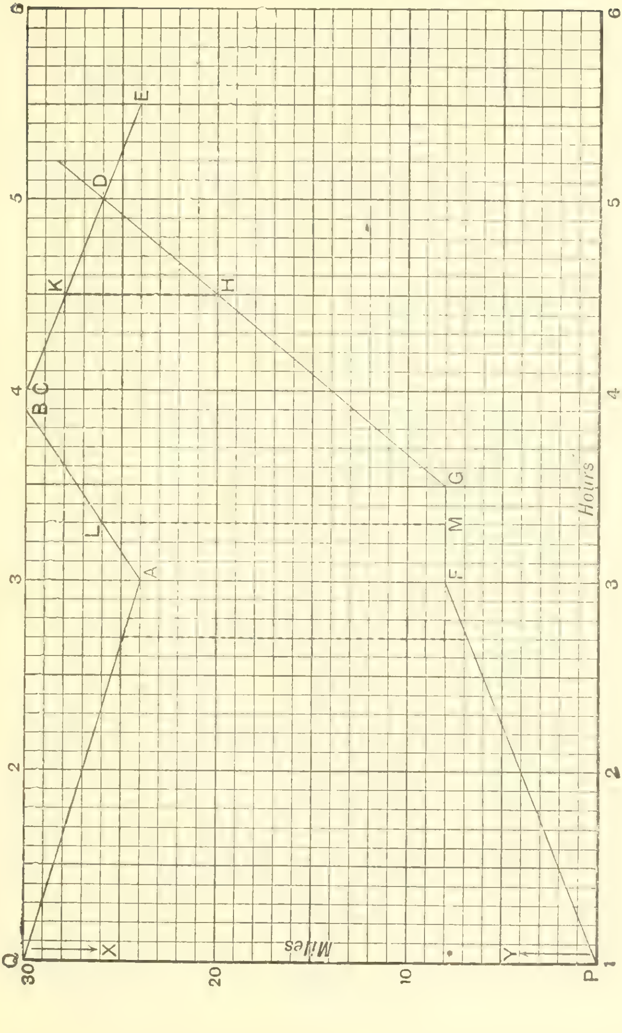
For Y's graph measure 0·8 inch vertically above 3 p.m. to F; then, since he walks 8 miles in 2 hours, PF is the first stage of the graph. The next half-hour is spent without advance towards Q; therefore the corresponding portion of the graph is FG.

GH represents the course of the bicycle ride at 12 miles an hour, and it will be found that it cuts X's graph at D.

Hence the point of meeting is at D, which is 4 miles from Q, and the time is 5 p.m. By inspection of the graphs we find LM and KH represent 18 and 8 miles respectively. The corresponding times are 3·3 and 4·5 hours; that is X and Y are 18 and 8 miles apart at 3.18 p.m. and 4.30 p.m. respectively.

They were also 18 mi. apart, approximately, at 18 min. before 3.

Note. The solution has here been given in full to illustrate and enforce the general principle on which the linear graphs depend. Solutions may usually be presented with less detail, and the results quickly obtained from a well-drawn diagram on a suitable scale.


Miles
Hours
Q
P
30
20
10
1
2
3
4
5
6
A
B
C
D
E
F
G
H
K
L
M
X
Y

EXAMPLE 2. *A, B, and C run a race of 300 yards. A and C start from scratch, and A covers the distance in 40 seconds, beating C by 60 yards. B, with 12 yards' start, beats A by 4 seconds. Supposing the rates of running in each case to be uniform, find graphically the relative positions of the runners when B passes the winning post. Find also by how many yards B is ahead of A when the latter has run three-fourths of the course.*

In Fig. 22 let time be measured horizontally (0·5 inch to 10 seconds), and distance vertically (1 inch to 60 yards). O is the starting point for A and C; take OP equal to 0·2 inch, representing 12 yards, on the vertical axis; then P is B's starting point.

A's graph is drawn by joining O to the point which marks 40 seconds. From this point measure a vertical distance of 1 inch downwards to Q. Then since 1 inch represents 60 yards, Q is C's position when A is at the winning post, and OQ is C's graph.

Along the time-axis take 1·8 inch to R, representing 36 seconds; then PR is B's graph.

Through R draw a vertical line to meet the graphs of A and C in S and T respectively. Then S and T mark the positions of A and C when B passes the winning post.

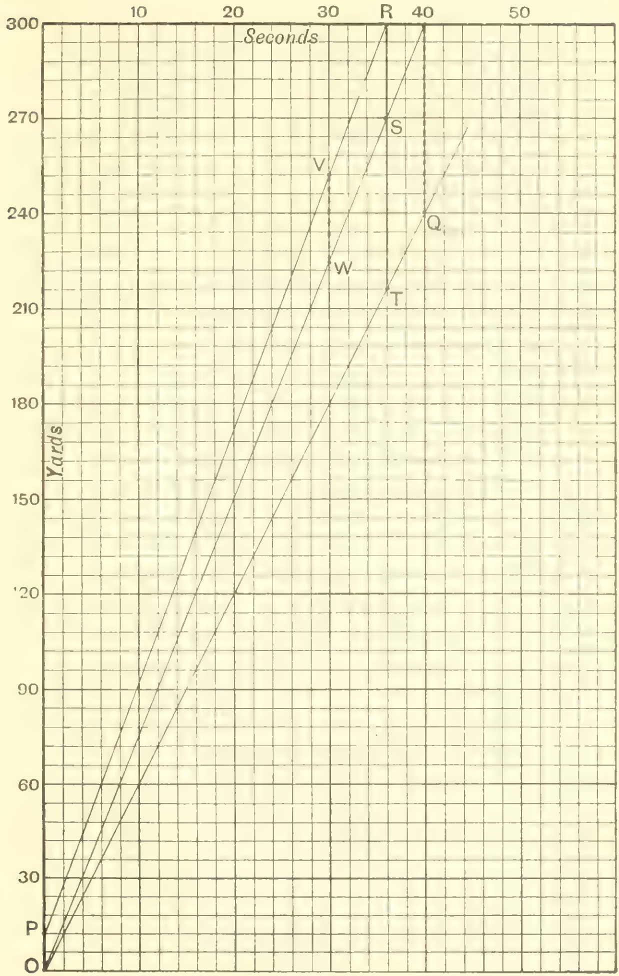
By inspection RS and ST represent 30 and 54 yards respectively.

Thus B is 30 yards ahead of A, and A is 54 yards ahead of C.

Again, since A runs three-fourths of the course in 30 seconds, the difference of the corresponding ordinates of A's and B's graphs after 30 seconds will give the distance between A and B. By measurement we find VW = 0·45 inch, which represents 27 yards.

The student is recommended to draw a figure for himself on a scale twice as large as that given in Fig. 22.

337. When a variable quantity y is partly constant and partly proportional to a variable quantity x, the algebraical relation between x and y is of the form $y = ax + b$, where a and b are constant. The corresponding graph will therefore be a straight line; and since a straight line is completely determined when the positicns of two points are known, it follows that, in all problems which can be illustrated by linear graphs, it is sufficient if the data furnish for each graph two independent pairs of simultaneous values of the variable quantities.



10
20
30
R
40
50
Seconds
300
270
240
210
180
150
120
90
60
30
P
O
Yards
V
S
Q
W
T

FIG. 22.

EXAMPLES XXXV. h.

1. At noon A starts to walk at 6 miles an hour, and at 1.30 p.m. B follows on horseback at 8 miles an hour. When will B overtake A? Also find

(i) when A is 5 miles ahead of B;

(ii) when A is 3 miles behind B.

[Take 1 inch horizontally to represent 1 hour, and 1 inch vertically to represent 10 miles.]

2. By measuring time along OX (1 inch for 1 hour) and distance along OY (1 inch for 10 miles) shew how to draw lines

(i) from O to indicate distance travelled towards Y at 12 miles an hour;

(ii) from Y to indicate distance travelled towards O at 9 miles an hour.

If these are the rates of two men, starting at noon to ride towards each other from two places 60 miles apart, find from the graphs when they are first 18 miles from each other. Also find (to the nearest minute) their time of meeting.

3. Two bicyclists ride to meet each other from two places 95 miles apart. A starts at 8 a.m. at 10 miles an hour, and B starts at 9.30 a.m. at 15 miles an hour. Find graphically when and where they meet, and at what times they are $37\frac{1}{2}$ miles apart.

4. A can beat B by 20 yards in 120, and B can beat C by 10 yards in 50. Supposing their rates of running to be uniform, find graphically how much start A can give C in 120 yards so as to run a dead heat with him. If A, B, and C start together, where are A and C when B has run 80 yards?

5. A, B, and C run a race of 200 yards. A gives B a start of 8 yards, and C starts some seconds after A. A runs the distance in 25 seconds and beats C by 40 yards. B beats A by 1 second, and when he has been running 15 seconds, he is 48 yards ahead of C. Find graphically how many seconds C starts after A. Shew also from the graphs that if the three runners started level they would run a dead heat. [Take 1 inch to 40 yards, and 1 inch to 10 seconds.]

6. A cyclist has to ride 75 miles. He rides for a time at 9 miles an hour and then alters his speed to 15 miles an hour covering the distance in 7 hours. At what time did he change his speed?

7. A and B ride to meet each other from two towns X and Y which are 60 miles apart. A starts at 1 p.m., and B starts 36 minutes later. If they meet at 4 p.m., and A gets to Y at 6 p.m., find the time when B gets to X. Also find the times when they are 22 miles apart. When A is half-way between X and Y, where is B?

8. At 8 a.m. A begins a ride on a motor car at 20 miles an hour, and an hour and a half later B, starting from the same point, follows on his bicycle at 10 miles an hour. After riding 36 miles, A rests for 1 hr. 24 min., then rides back at 9 miles an hour. Find graphically when and where he meets B. Also find (i) at what time the riders were 21 miles apart, (ii) how far B will have ridden by the time A gets back to his starting point.

9. I row against a stream flowing $1\frac{1}{2}$ miles an hour to a certain point, and then turn back, stopping two miles short of the place whence I originally started. If the whole time occupied in rowing is 2 hrs. 10 mins. and my uniform speed in still water is $4\frac{1}{2}$ miles an hour, find graphically how far upstream I went.

[Take 1·2 of an inch horizontally to represent 1 hour, and 1 inch to 2 miles vertically.]

10. A boy starts from home and walks to school at the rate of 10 yards in 3 seconds, and is 20 seconds too soon. The next day he walks at the rate of 40 yards in 17 seconds, and is half a minute late. Find graphically the distance to the school, and shew that he would have been just in time if he had walked at the rate of 20 yards in 7 seconds.

11. Taking 1 inch as unit for x, and 0·5 as unit for y, draw the graph of $y=x^2$, and employ it to find the squares of 0·72, 1·7, 3·4; and the square roots of 7·56, 5·29, 9·61.

12. Draw the graph of $y=\sqrt{x}$ taking the unit for y five times as great as that for x.

By means of this curve check the values of the square roots found in Ex. 11.

13. Draw a graph which will give the square roots of all numbers between 25 and 36, to three places of decimals.

[Take the origin at the point 5, 25, and plot the graph of $y=x^2$; use for units of x and y, 10 inches and 0·5 inch respectively.]

14. From the graph of $y=x^3$ (on the scale of the diagram of Example 13) find the values of $\sqrt[3]{9}$ and $\sqrt[3]{9\cdot8}$ to 4 significant figures.

MISCELLANEOUS EXAMPLES VI.

1. Simplify $b-\{b-(a+b)-[b-(b-\overline{a-b})]+2a\}$.

2. Find the sum of

$$a+b-2(c+d),\ \ b+c-3(d+a),\ \text{and}\ \ c+d-4(a+b).$$

3. Multiply $\frac{1}{2}x+\frac{2}{3}y$ by $x-\frac{1}{3}y$.

4. If $x=6$, $y=4$, $z=3$, find the value of $\sqrt[3]{2x+3y+z}$.

5. Find the square of $2-3x+x^2$.

6. Solve $\frac{x+3}{x-1}+\frac{x-4}{x-6}=2$.

7. Find the H.C.F. of a^3-2a-4 and a^3-a^2-4.

8. Simplify $\frac{2a}{a+b}+\frac{2b}{a-b}-\frac{a^2+b^2}{a^2-b^2}$.

9. Solve

$$\left.\begin{aligned}\frac{3}{5}x+\frac{y}{4}&=13\\ \frac{1}{3}x-\frac{y}{8}&=\ 3\end{aligned}\right\}.$$

10. Two digits, which form a number, change places when 18 is added to the number, and the sum of the two numbers thus formed is 44: find the digits.

11. If $a=1$, $b=-2$, $c=3$, $d=-4$, find the value of

$$\frac{a^2b^2+b^2c+d(a-b)}{10a-(c+b)^2}.$$

12. Subtract $-x^2+y^2-z^2$ from the sum of

$$\frac{1}{3}x^2+\frac{1}{4}y^2,\ \ \frac{1}{5}y^2+\frac{1}{3}z^2,\ \ \text{and}\ \ \frac{1}{3}z^2-\frac{1}{4}x^2.$$

13. Write down the cube of $x+8y$.

14. Simplify $\frac{x^2+xy}{x^2+y^2}\times\frac{x^4-y^4}{xy+y^2}\times\frac{y}{x}$.

15. Solve $\frac{3}{5}(2x-7)-\frac{2}{3}(x-8)=\frac{4x+1}{15}+4$.

16. Find the H.C.F. and L.C.M. of

$$x^4+x^3+2x-4\ \ \text{and}\ \ x^3+3x^2-4.$$

17. Find the square root of $4a^4+9(1-2a)+3a^2(7-4a)$.

18. Solve
$$\left.\begin{aligned} y&=\frac{x+a}{2}+\frac{b}{3} \\ x&=\frac{y+b}{2}+\frac{a}{3} \end{aligned}\right\}.$$

19. Simplify $\left(\dfrac{a}{x+a}-\dfrac{x}{x-a}\right)\div\dfrac{x^2+a^2}{x^2+ax}$.

20. When 1 is added to the numerator and denominator of a certain fraction the result is equal to $\frac{3}{2}$; and when 1 is subtracted from its numerator and denominator the result is equal to 2: find the fraction.

21. Shew that the sum of $12a+6b-c$, $-7a-b+c$, and $a+b+6c$, is six times the sum of $25a+13b-8c$, $-13a-13b-c$, and $-11a+b+10c$.

22. Divide $x^2-xy+\frac{3}{16}y^2$ by $x-\frac{1}{4}y$.

23. Add together $18\left\{\dfrac{2x}{9}-\dfrac{1}{6}\left(\dfrac{2y}{3}+z\right)\right\}$,

$$24\left(\frac{3x}{8}-\frac{2y-3z}{12}\right), \text{ and } 30\left\{\frac{7z}{15}-\frac{4}{5}(2x-y)\right\}.$$

24. Find the factors of

(1) $10x^2+79x-8$. (2) $729x^6-y^6$.

25. Solve $\dfrac{2x-1}{5}+\dfrac{5x+3}{17}=3-\dfrac{4x-118}{11}$.

26. Find the value of
$$(5a-3b)(a-b)-b\{3a-c(4a-b)-b^2(a+c)\},$$
when $a=0$, $b=-1$, $c=\frac{1}{2}$.

27. Find the H.C.F. of
$$7x^3-10x^2-7x+10 \text{ and } 2x^3-x^2-2x+1.$$

28. Simplify $\dfrac{x^2-7xy+12y^2}{x^2+5xy+6y^2}\div\dfrac{x^2-5xy+4y^2}{x^2+xy-2y^2}$.

29. Solve
$$\left.\begin{aligned} 3abx+y&=9b \\ 4abx+3y&=17b \end{aligned}\right\}.$$

30. Find the two times between 7 and 8 o'clock when the hands of a watch are separated by 15 minutes.

31. If $a=1$, $b=-2$, $c=3$, $d=-4$, find the value of

$$\sqrt{d^2-4b+a^2}-\sqrt{c^3+b^3+a+d}.$$

32. Multiply the product of $\frac{1}{4}x^2-\frac{1}{2}xy+y^2$ and $\frac{1}{2}x+y$ by x^3-8y^3.

33. Simplify by removing brackets

$$a^4-\{4a^3-(6a^2-4a+1)\}$$
$$-[-2-\{a^4-(-4a^3-\overline{6a^2-4a})\}-(8a-1)].$$

34. Find the remainder when $5x^4-7x^3+3x^2-x+8$ is divided by $x-4$.

35. Simplify $\frac{x^2+y^2}{x^2-xy}\times\frac{xy-y^2}{x^4-y^4}\times\frac{x}{y}$.

36. Solve

$$\left.\begin{aligned}\frac{x-11}{3}+y&=18\\ 2x+\frac{y-13}{4}&=29\end{aligned}\right\}.$$

37. Find the square root of $4x^6-12x^4+28x^3+9x^2-42x+49$.

38. Solve $\cdot006x-\cdot491+\cdot723x=-\cdot005$.

39. Find the L.C.M. of x^3+y^3, $3x^2+2xy-y^2$, and $x^3-x^2y+xy^2$.

40. A bill of \$12.50 is paid with quarters and half-dollars, and twice the number of half-dollars exceeds three times that of the quarters by 10: how many of each are used?

41. Simplify

$$(a+b+c)^2-(a-b+c)^2+(a+b-c)^2-(-a+b+c)^2.$$

42. Find the remainder when $a^4-3a^3b+2a^2b^2-b^4$ is divided by $a^2-ab+2b^2$.

43. If $a=0$, $b=1$, $c=-2$, $d=3$, find the value of

$$(3abc-2bcd)\sqrt[3]{a^3bc-c^3bd+3}.$$

44. Find an expression which will divide both $4x^2+3x-10$ and $4x^3+7x^2-3x-15$ without remainder.

45. Simplify $\dfrac{a+\dfrac{ab}{a-b}}{a^2-\dfrac{2a^2b^2}{a^2+b^2}}\times\dfrac{\dfrac{1}{a^2}-\dfrac{1}{b^2}}{\dfrac{1}{a}-\dfrac{1}{b}}$.

46. Find the cube root of $8x^3-2x^2y+\frac{xy^2}{6}-\frac{y^3}{216}$.

47. Solve

$$\left.\begin{aligned}9x+8y&=43xy\\ 8x+9y&=42xy\end{aligned}\right\}.$$

48. Simplify $\dfrac{3}{x-4}-\dfrac{2}{x-5}-\dfrac{x-7}{(x-2)(x-3)}$.

49. Find the L.C.M. of $8x^3+38x^2+59x+30$
and $6x^3-13x^2-13x+30$.

50. A boy spent half of his money in one shop, one third of the remainder in a second, and one-fifth of what he had left in a third. He had 20 cents at last: how much had he at first?

51. Find the remainder when $x^7-10x^6+8x^5-7x^3+3x-11$ is divided by x^2-5x+4.

52. Simplify $4\left\{a-\dfrac{3}{2}\left(b-\dfrac{4c}{3}\right)\right\}\left\{\dfrac{1}{2}(2a-b)+2(b-c)\right\}$.

53. If $a=\dfrac{25}{16}$, $b=1$, $c=\dfrac{3}{4}$, prove that

$$(a-\sqrt{b})(\sqrt{a}+b)\sqrt{a-b}=\frac{3c^4}{\sqrt{a-c^2}}.$$

54. Find the L.C.M. of $x^2-7x+12$, $3x^2-6x-9$, and $2x^2-6x-8$.

55. Find the sum of the squares of $ax+by$, $bx-ay$, $ay+bx$, $by-ax$; and express the result in factors.

56. Solve $\dfrac{x}{6}+\dfrac{y}{4}=\dfrac{3x-5z}{4}=\dfrac{z}{8}+\dfrac{7y}{16}=1$.

57. Simplify $\dfrac{a^3+b^3}{a^4-b^4}-\dfrac{a+b}{a^2-b^2}-\dfrac{1}{2}\left\{\dfrac{a-b}{a^2+b^2}-\dfrac{1}{a-b}\right\}$.

58. Solve $x-\left(3x-\dfrac{2x+5}{10}\right)=\dfrac{1}{6}(2x+67)+\dfrac{5}{3}\left(1+\dfrac{x}{5}\right)$.

59. Add together the following fractions:

$$\frac{2}{x^2+xy+y^2},\quad \frac{-4x}{x^3-y^3},\quad \frac{x^2}{y^2(x-y)^2},\quad \frac{-x^2}{x^3y-y^4}.$$

60. A man agreed to work for 30 days, on condition that for every day's work he should receive \$2.50, and that for every day's absence from work he should forfeit \$1.50; at the end of the time he received \$51: how many days did he work?

61. Divide $\dfrac{3x^5}{4}+27-\dfrac{43x^2}{4}-4x^4+\dfrac{77x^3}{8}-\dfrac{33x}{4}$ by $\dfrac{x^2}{2}+3-x$.

62. Find the value of

$$\frac{4y}{5}(y-x)-35\left[\frac{3x-4y}{5}-\frac{1}{10}\left\{3x-\frac{5}{7}(7x-4y)\right\}\right]$$

when $x=-\dfrac{1}{2}$ and $y=2$.

63. Simplify $\dfrac{10x-11}{3(x^2-1)}-\dfrac{10x-1}{3(x^2+x+1)}+\dfrac{x^2-2x+5}{(x^3-1)(x+1)}$.

64. Find the cube root of $\dfrac{a^3c^3}{b^3}x^6-\dfrac{3a^2c}{b}x^5+\dfrac{3ab}{c}x^4-\dfrac{b^3}{c^3}x^3$.

65. Solve $\dfrac{4x-17}{x-4}+\dfrac{10x-13}{2x-3}=\dfrac{8x-30}{2x-7}+\dfrac{5x-4}{x-1}$.

66. Find the factors of

(1) x^3+5x^2+x+5. (2) $x^2-2xy-323y^2$.

67. Solve

$$\left.\begin{aligned}\frac{1}{3}(x+y)+2z&=21\\ 3x-\frac{1}{2}(y+z)&=65\\ x+\frac{1}{2}(x+y-z)&=38\end{aligned}\right\}.$$

68. Simplify $\dfrac{x+2y}{\frac{2}{7}x-y}-\dfrac{3x^2+63xy+70y^2}{2x^2+3xy-35y^2}$.

69. Find the square root of $-(3b-2c-2a)^3\{2(a+c)-3b\}$.

70. The united ages of a man and his wife are six times the united ages of their children. Two years ago their united ages were ten times the united ages of their children, and six years hence their united ages will be three times the united ages of the children. How many children have they?

71. Find the sum of

$x^2-3xy-\frac{2}{3}y^2$, $2y^2-\frac{2}{3}y^3+z^2$, $xy-\frac{1}{3}y^2+y^3$, and $2xy-\frac{1}{3}y^3$.

72. From $\{(a+b)(a-x)-(a-b)(b-x)\}$ subtract $(a+b)^2-2bx$.

73. If $a=5$, $b=4$, $c=3$, find the value of

$$\sqrt[3]{6abc+(b+c)^3+(c+a)^3+(a+b)^3-(a+b+c)^3}.$$

74. Find the factors of

(1) $3x^3+6x^2-189x$. (2) $a^2+2ab+b^2+a+b$.

75. Solve

$$\left.\begin{aligned}px&=qy\\ (p+q)x-(q-p)y&=r\end{aligned}\right\}.$$

76. Simplify $\dfrac{x+\dfrac{y}{2}}{2x^2+xy+\dfrac{y^2}{2}}-\dfrac{x^2-\dfrac{y^2}{2}}{4\left(x^3-\dfrac{y^3}{8}\right)}$.

77. Solve $\dfrac{x-7}{x+7}+\dfrac{1}{2(x+7)}=\dfrac{2x-15}{2x-6}$.

78. Reduce $\dfrac{x^4-x^2-2x+2}{2x^3-x-1}$ to its lowest terms.

79. Add together the fractions:

$$\frac{1}{2x^2-4x+2},\quad \frac{1}{2x^2+4x+2},\quad \text{and}\quad \frac{1}{1-x^2}.$$

80. A number consists of three digits, the right-hand one being zero. If the left-hand and middle digits be interchanged the number is diminished by 180; if the left-hand digit be halved, and the middle and right-hand digit be interchanged, the number is diminished by 336: find the number.

81. Divide $1-5x+\dfrac{152}{15}x^3-\dfrac{106}{225}x^4-\dfrac{28}{9}x^5$ by $1-x-\dfrac{14}{15}x^2$.

82. If $p=1$, $q=\dfrac{1}{2}$, find the value of

$$\frac{(p^2+q^2)-(p-q)\sqrt{p^2+2pq+q^2}}{2p+q-\{p-(q-p)\}}.$$

83. Multiply $\dfrac{3x^3}{2}-5x^2+\dfrac{x}{4}+9$ by $\dfrac{x^2}{2}-x+3$.

84. Find the L.C.M. of

$$(a^2b-2ab^2)^2,\quad 2a^2-3ab-2b^2,\quad \text{and}\quad 2(2a^2+ab)^2.$$

85. Solve $\dfrac{2x+3}{x+1}=\dfrac{4x+5}{4x+4}+\dfrac{3x+3}{3x+1}$.

86. Reduce $\dfrac{5x^3-14x^2+16}{3x^3-2x^2+16x-48}$ to its lowest terms.

87. Find the square root of

$$4a^4+9\left(a^2+\frac{1}{a^2}\right)+12a(a^2+1)+18.$$

88. Solve

$$\left.\begin{aligned}\frac{x}{2a}+\frac{y}{3b}&=a+b\\ \frac{3x}{a}-\frac{2y}{b}&=6(b-a)\end{aligned}\right\}.$$

89. Multiply

$$3x+4y+\frac{11xy}{x-\frac{3}{2}y}\ \text{ by }\ 10x-3y-\frac{11xy}{\frac{x}{4}+y}.$$

90. A bag contained \$10 in quarters and ten-cent pieces; after 17 ten-cent pieces and 6 quarters were taken out, thrice as many quarters as ten cent pieces were left: find the number of each coin.

91. Find the value of

$$5(a-b)-2\{3a-(a+b)\}+7\{(a-2b)-(5a-2b)\},$$

when $a=-\frac{1}{9}b$.

92. Divide $3x^4-5x^3+7x^2-11x-13$ by $3x-2$.

93. Find the L.C.M. of

$15(p^3+q^3)$, $5(p^2-pq+q^2)$, $4(p^2+pq+q^2)$, and $6(p^2-q^2)$.

94. Resolve into factors:

(1) a^3-8b^{15}. (2) $-x^2+2x-1+x^4$.

95. Solve $\dfrac{x+a}{x+b}=\dfrac{x+3a}{x+a+b}$.

96. Simplify

(1) $\dfrac{35a^2b^2c^2-49b^3c^3}{65a^5bc-91a^3b^2c^2}$. (2) $\dfrac{y^4-7y^3+8y^2-12y}{2y^2-2y-60}$.

97. Solve

$$\left.\begin{aligned} 7x-9y+4z&=16\\ \frac{x+y}{3}&=\frac{x+y+z}{2}\\ 2x-3y+4z-5&=0 \end{aligned}\right\}.$$

98. Simplify $\dfrac{y^2-\dfrac{2y}{y-1}}{y^2-\dfrac{2y}{y+1}}\div\left(\dfrac{y^2-5y-6}{y^2-6y+5}\times\dfrac{y-2}{y+2}\right)$.

99. Find the square root of

$$\frac{4a^2-12ab-6bc+4ac+9b^2+c^2}{4a^2+9c^2-12ac}.$$

100. The express leaves Bristol at 3 p.m. and reaches London at 6; the ordinary train leaves London at 1.30 p.m. and arrives at Bristol at 6. If both trains travel uniformly, find the time when they will meet.

101. Solve (1) $\cdot\dot{6}x+\cdot75x-\cdot1\dot{6}=x-\cdot58\dot{3}x+5$.

(2) $\dfrac{37}{x^2-5x+6}+\dfrac{4}{x-2}=\dfrac{7}{3-x}$.

102. Simplify (1) $\dfrac{a+x}{a^2+ax+x^2}+\dfrac{a-x}{a^2-ax+x^2}+\dfrac{2x^3}{a^4+a^2x^2+x^4}$.

$$(2)\ (1+x)^2 \div \left\{1+\cfrac{x}{1-x+\cfrac{x}{1+x+x^2}}\right\}.$$

103. Find the square root of

$$a^6+\frac{1}{a^6}-6\left(a^4+\frac{1}{a^4}\right)+15\left(a^2+\frac{1}{a^2}\right)-20;$$

also the cube root of the result.

104. Divide $1-2x$ by $1+3x$ to 4 terms.

105. I bought a horse and carriage for \$450; I sold the horse at a gain of 5 per cent., and the carriage at a gain of 20 per cent., making on the whole a gain of 10 per cent. Find the original cost of the horse.

106. Find the divisor when $(4a^2+7ab+5b^2)^2$ is the dividend, $8(a+2b)^2$ the quotient, and $b^2(9a+11b)^2$ the remainder.

107. Solve (1) $5x(x-3)=2(x-7)$.

$$(2)\ \frac{1}{(x-1)(x-2)}+6=\frac{3}{x-2}+\frac{2}{x-1}.$$

108. If $x=a+b+\dfrac{(a-b)^2}{4(a+b)}$, and $y=\dfrac{a+b}{4}+\dfrac{ab}{a+b}$.

prove that $(x-a)^2-(y-b)^2=b^2$.

109. Find the square root of

$$49x^4+\frac{1051x^2}{25}-\frac{14x^3}{5}-\frac{6x}{5}+9.$$

110. Solve $\dfrac{a+x}{a^2+ax+x^2}+\dfrac{a-x}{a^2-ax+x^2}=\dfrac{3a}{x(a^4+a^2x^2+x^4)}$.

111. Subtract $\dfrac{x+3}{x^2+x-12}$ from $\dfrac{x+4}{x^2-x-12}$,

and divide the difference by $1+\dfrac{2(x^2-12)}{x^2+7x+12}$.

112. Find the H.C.F. and L.C.M. of

$2x^2+(6a-10b)x-30ab$ and $3x^2-(9a+15b)x+45ab$.

113. Solve (1) $2cx^2-abx+2abd=4cdx$.

$$(2)\ \frac{x}{2(x+3)}-2\tfrac{5}{24}=\frac{x^2}{x^2-9}-\frac{8x-1}{4(x-3)}.$$

114. If $a=1$, $b=2$, $c=3$, $d=4$, find the value of

$$\frac{a^b+b^c+c^d}{b^a+c^b+d^c+(a+b)(b+c)}+3(a^a+b^b+c^c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).$$

115. I rode one-third of a journey at 10 miles an hour, one-third more at 9, and the rest at 8 miles an hour; if I had ridden half the journey at 10, and the other half at 8 miles per hour, I should have been half a minute longer on the way: what distance did I ride?

116. The product of two factors is $(3x+2y)^3-(2x+3y)^3$, and one of the factors is $x-y$; find the other factor.

117. If $a+b=1$, prove that $(a^2-b^2)^2=a^3+b^3-ab$.

118. Resolve into factors:

(1) $x^3+y^3+3xy(x+y)$. (2) $m^3-n^3-m(m^2-n^2)+n(m-n)^2$.

119. Solve (1) $\left.\begin{aligned} x^3-y^3&=28 \\ x^2+xy+y^2&=7 \end{aligned}\right\}$. (2) $\left.\begin{aligned} x^2-6xy+11y^2&=9 \\ x-3y&=1 \end{aligned}\right\}$.

120. Find the square root of

$$(a-b)^4-2(a^2+b^2)(a-b)^2+2(a^4+b^4).$$

121. Simplify the fractions:

(1) $\dfrac{1}{a^2-\dfrac{a^3-1}{a+\dfrac{1}{a+1}}}$. (2) $\dfrac{\left(1+\dfrac{1}{x}\right)\times\left(1-\dfrac{1}{x}\right)^2}{x-\dfrac{1}{x}}$.

122. Find the H.C.F. of

$$a^2b+b^2c-abc-ab^2 \text{ and } ax^2+ab-a^2-bx^2.$$

123. A constituency had two-thirds of its number Conservatives: in an election 25 refused to vote, and 60 went over to the Liberals; the voters were now equal. How many voters were there altogether?

124. Solve (1) $\dfrac{x^2}{a+b}+(a-b)=\dfrac{2ax}{a+b}$.

(2) $\dfrac{3}{x}+\dfrac{2}{y}=6\left(\dfrac{1}{y}-\dfrac{1}{2x}\right)=2$.

125. Simplify (1) $\left(1+\dfrac{y^2+z^2-x^2}{2yz}\right)\div\left(1-\dfrac{x^2+y^2-z^2}{2xy}\right)$.

(2) $\dfrac{(x+1)^3-(x-1)^3}{(x+1)^4-(x-1)^4}$.

126. Divide

$$x^4+(a-1)x^3-(2a+1)x^2+(a^2+4a-5)x+3a+6$$

by $$x^2-3x+a+2.$$

127. Resolve into factors:

(1) $x^2+5xy-24y^2+x-3y$. (2) $x^3-\dfrac{4}{x}$.

128. Find the square root of p^2-3q to three terms.

129. Solve (1) $\dfrac{x-5}{x-6}-\dfrac{x-6}{x-7}=\dfrac{x-1}{x-2}-\dfrac{x-2}{x-3}$.

(2) $ax+1=by+1=ay+bx$.

130. Find the H.C.F. of $3x^2+(4a-2b)x-2ab+a^2$ and

$$x^3+(2a-b)x^2-(2ab-a^2)x-a^2b.$$

131. Simplify

(1) $\dfrac{(x^a)^3}{x^{b+c}}\times\dfrac{(x^b)^3}{x^{c+a}}\times\dfrac{(x^c)^3}{x^{a+b}}$. (2) $x^{\frac{1}{2}}y^{\frac{1}{3}}\left(\dfrac{y^{\frac{1}{4}}}{x^{\frac{1}{6}}}\right)^2\div\dfrac{y^{-\frac{1}{4}}}{x^{\frac{1}{4}}}$.

132. At a cricket match the contractor provided dinner for 27 persons, and fixed the price so as to gain $12\frac{1}{2}$ per cent. upon his outlay. Six of the cricketers being absent, the remaining 21 paid the fixed price for their dinner, and the contractor lost \$3: what was the charge for the dinner?

133. Prove that $x(y+2)+\dfrac{x}{y}+\dfrac{y}{x}$ is equal to a, if

$$x=\frac{y}{y+1} \text{ and } y=\frac{a-2}{2}.$$

134. Find the cube root of

$$x^3-12x^2+54x-112+\frac{108}{x}-\frac{48}{x^2}+\frac{8}{x^3}.$$

135. Find the H.C.F. and L.C.M. of

$$x^3+2ax^2+a^2x+2a^3 \text{ and } x^3-2ax^2+a^2x-2a^3.$$

136. Simplify

(1) $42\left\{\dfrac{4x-3y}{6}-\dfrac{3x-4y}{7}\right\}-56\left\{\dfrac{3x-2y}{7}-\dfrac{2x-3y}{8}\right\}$,

(2) $\dfrac{4b+a}{3b+a}+\dfrac{a-4b}{a-3b}+\dfrac{a^2-3b^2}{a^2-9b^2}$.

137. Resolve $4a^2(x^3+18ab^2)-(32a^5+9b^2x^3)$ into four factors.

138. Solve (1) $5\sqrt{3x}-1=\sqrt{75x-29}$.

(2) $\dfrac{xy}{x+y}=70,\quad \dfrac{xz}{x+z}=84,\quad \dfrac{yz}{y+z}=140.$

139. Shew that the difference between

$$\frac{x}{x-a}+\frac{x}{x-b}+\frac{x}{x-c} \text{ and } \frac{a}{x-a}+\frac{b}{x-b}+\frac{c}{x-c}$$

is the same whatever value x may have.

140. Multiply $x^{\frac{3}{2}}+2y^{\frac{3}{2}}+3z^{\frac{3}{2}}$ by $x^{\frac{3}{2}}-2y^{\frac{3}{2}}-3z^{\frac{3}{2}}$.

141. Walking $4\frac{1}{4}$ miles an hour, I start $1\frac{1}{2}$ hours after a friend whose pace is 3 miles an hour: how long shall I be in overtaking him?

142. Express in the simplest form

(1) $(8^{\frac{2}{3}}+4^{\frac{3}{2}})\times 16^{-\frac{3}{4}}$. (2) $\dfrac{\left\{9^n \,.\, 3^2\times\dfrac{1}{3^{-n}}\right\}-27^n}{3^{3n}\times 9}$.

143. Find the square root of

$$\frac{x}{y}+\frac{y}{x}+3-2\sqrt{\frac{x}{y}}-2\sqrt{\frac{y}{x}}.$$

144. Simplify

(1) $\left(\dfrac{x}{x-1}-\dfrac{1}{x+1}\right)\cdot\dfrac{x^3-1}{x^6+1}\cdot\dfrac{(x-1)^2(x+1)^2+x^2}{x^4+x^2+1}$.

(2) $\left\{\dfrac{a^4-y^4}{a^2-2ay+y^2}\div\dfrac{a^2+ay}{a-y}\right\}\times\left\{\dfrac{a^5-a^3y^2}{a^3+y^3}\div\dfrac{a^4-2a^3y+a^2y^2}{a^2-ay+y^2}\right\}$.

145. Find the value of

(1) $\sqrt{8}+\sqrt{50}-\sqrt{18}+\sqrt{48}$. (2) $\sqrt{35+14\sqrt{6}}$.

146. Solve (1) $\dfrac{x-b}{x-a}-\dfrac{x-a}{x-b}=\dfrac{2(a-b)}{x-(a+b)}$.

(2) $\left.\begin{aligned} 2x+3y &= 1\tfrac{1}{2} \\ 4x^2+9xy+9y^2 &= 11 \end{aligned}\right\}$.

147. Shew that

$$\frac{(a+b)^3-c^3}{(a+b)-c}+\frac{(b+c)^3-a^3}{b+c-a}+\frac{(c+a)^3-b^3}{c+a-b}$$

is equal to $2(a+b+c)^2+a^2+b^2+c^2$.

148. Divide $a-x+4a^{\frac{1}{4}}x^{\frac{3}{4}}-4a^{\frac{1}{2}}x^{\frac{1}{2}}$

by $a^{\frac{1}{2}}+2a^{\frac{1}{4}}x^{\frac{1}{4}}-x^{\frac{1}{2}}$.

149. Find the square root of

$$(a-1)^4+2(a^4+1)-2(a^2+1)(a-1)^2.$$

150. How much are pears a gross when 12 more for a dollar lowers the price five cents a dozen?

151. Shew that if a number of two digits is six times the sum of its digits, the number formed by interchanging the digits is five times their sum.

152. Find the value of

$$\frac{1}{(a-b)(b-c)}-\frac{1}{(b-c)(a-c)}-\frac{1}{(c-a)(b-a)}.$$

153. Multiply

$$3+5x-\frac{12+41x+36x^2}{4+7x} \text{ by } 5-2x+\frac{26x-8x^2-14}{3-4x}.$$

154. If $x-\dfrac{1}{x}=1$, prove that $x^2+\dfrac{1}{x^2}=3$, and $x^3-\dfrac{1}{x^3}=4$.

155. Solve (1) $\dfrac{3x}{11}+\dfrac{23}{x+4}=\dfrac{1}{3}(x+5)$.

(2) $\left.\begin{aligned} 2x^2-3y^2&=23 \\ 2xy-3y^2&=3 \end{aligned}\right\}$.

156. Simplify

(1) $1\frac{3}{5}\sqrt{20}-3\sqrt{5}-\sqrt{\dfrac{1}{5}}$. (2) $\dfrac{\sqrt{x}}{y^{-\frac{1}{3}}}\left(\dfrac{\sqrt[4]{y}}{x^{\frac{1}{6}}}\right)\div\dfrac{y^{-\frac{1}{4}}}{x^{\frac{1}{4}}}$.

157. Find the H.C.F. of $(p^2-1)x^2+(3p-1)x-p(p-1)$ and

$$p(p+1)x^2-(p^2-2p-1)x-(p-1).$$

158. Reduce to its simplest form

$$\frac{ax+\dfrac{a}{y}}{x-\dfrac{1}{y}}\times\frac{x^2+\dfrac{1}{y^2}}{bx^2-\dfrac{b}{y^2}}\times\frac{\dfrac{1}{5}(xy-1)^2}{\dfrac{1}{3}(x^4y^4-1)}.$$

159. Find the square root of

(1) $1-2^{2n+1}+4^{2n}$. (2) $9^n-2\,.\,6^n+4^n$.

160. A clock gains 4 minutes a day. What time should it indicate at 6 o'clock in the morning, in order that it may be right at 7.15 p.m. on the same day?

161. If $x=2+\sqrt{2}$, find the value of $x^2+\dfrac{4}{x^2}$.

162. Solve

(1) $\dfrac{\sqrt{x+a}}{\sqrt{x-b}}=\dfrac{\sqrt{x-a}}{\sqrt{x}}$. (2) $\dfrac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}=3$.

163. Simplify

$$\frac{a^2}{(b-a)(c-a)}+\frac{b^2}{(c-b)(a-b)}+\frac{c^2}{(a-c)(b-c)}.$$

164. Find the product of $\frac{1}{5}\sqrt{5}$, $\frac{1}{2}\sqrt[3]{2}$, $\sqrt[6]{80}$, $\sqrt[3]{5}$, and divide

$$\frac{8-4\sqrt{5}}{\sqrt{5}+1} \text{ by } \frac{3\sqrt{5}-7}{5+\sqrt{7}}.$$

165. Resolve $9x^6y^2-576y^2-4x^8+256x^2$ into six factors.

166. Simplify

(1) $\dfrac{1-\dfrac{a^2}{(x+a)^2}}{(x+a)(x-a)}\div\dfrac{x(x+2a)}{(x^2-a^2)(x+a)^2}$.

(2) $\dfrac{6x^2y^2}{m+n}\div\left[\dfrac{3(m-n)x}{7(r+s)}\div\left\{\dfrac{4(r-s)}{21xy^2}\div\dfrac{r^2-s^2}{4(m^2-n^2)}\right\}\right]$.

167. Simplify (1) $\left(a^{1+\frac{q}{p}}\right)^{\frac{p}{p+q}}\div\sqrt[p]{\dfrac{a^{2p}}{(a^{-1})^{-p}}}$.

(2) $\sqrt{14-\sqrt{132}}$.

168. Find the H.C.F. and L.C.M. of

$$20x^4+x^2-1,\ \ 25x^4+5x^3-x-1,\ \ 25x^4-10x^2+1.$$

169. Solve (1) $a+x+\sqrt{2ax+x^2}=b$.

(2) $x+9\frac{5}{8}+\dfrac{1}{\dfrac{x}{7}+\dfrac{11}{8}}=8$.

170. The price of photographs is raised \$3 per dozen, and customers consequently receive ten less than before for \$5: what were the prices charged?

171. If $\left(a+\dfrac{1}{a}\right)^2=3$, prove that $a^3+\dfrac{1}{a^3}=0$.

172. Find the value of

$$\frac{x+2a}{2b-x}+\frac{x-2a}{2b+x}+\frac{4ab}{x^2-4b^2},\text{ when } x=\frac{ab}{a+b}.$$

173. Reduce to fractions in their lowest terms

(1) $\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \div \left(\frac{x+y+z}{x^2+y^2+z^2-xy-yz-zx}-\frac{1}{x+y+z}\right)+1.$

(2) $\left(1-\frac{56}{x+4}+\frac{42}{x+3}\right)\left(1+\frac{56}{x-4}-\frac{42}{x-3}\right).$

174. Express as a whole number

$$(27)^{\frac{2}{3}}+(16)^{\frac{3}{4}}-\frac{2}{(8)^{-\frac{2}{3}}}+\frac{\sqrt[5]{2}}{(4)^{-\frac{2}{5}}}.$$

175. Simplify

(1) $\frac{n}{1-x^n}+\frac{n}{1-x^{-n}}$ (2) $\sqrt[4]{97-56\sqrt{3}}.$

176. Solve

(1) $\frac{x-4a}{x-3a}+\frac{x-5a}{x-4a}=\frac{x+6a}{x-4a}+\frac{x+5a}{x-3a}.$

(2) $\left.\begin{aligned} 3x^2+xy+3y^2&=8\tfrac{1}{4} \\ 8x^2-3xy+8y^2&=17\tfrac{3}{4} \end{aligned}\right\}.$

177. Find the square root of $\frac{a^2x^2+2ab^2x^3+b^4x^4}{a^{2m}+2a^mx^n+x^{2n}}.$

178. Simplify

(1) $\frac{b}{\sqrt{a}} \times \sqrt[3]{ac} \times \frac{\sqrt[4]{c^3}}{\sqrt{b}} \times \frac{\sqrt{b^{-1}}}{a^{-\frac{1}{6}}}.$ (2) $\left\{\frac{(9^{n+\frac{1}{4}}) \times \sqrt{3 \,.\, 3^n}}{3\sqrt{3^{-n}}}\right\}^{\frac{1}{n}}.$

179. A boat's crew can row 8 miles an hour in still water; what is the speed of a river's current if it take them 2 hours and 40 minutes to row 8 miles up and 8 miles down?

180. If $a=x^2-yz$, $b=y^2-zx$, $c=z^2-xy$, prove that

$$a^2-bc=x(ax+by+cz).$$

181. Find a quantity such that when it is subtracted from each of the quantities a, b, c, the remainders are in continued proportion.

182. Simplify

(1) $\left(x+y-\frac{1}{x+y-\frac{xy}{x+y}}\right) \times \frac{x^3-y^3}{x^2-y^2}.$

(2) $\frac{2(7x-4)}{6x^2-7x+2}+\frac{x-10}{6x^2-x-2}-\frac{2(4x-1)}{4x^2-1}.$

183. Find the sixth root of

$$729-2916x^2+4860x^4-4320x^6+2160x^8-576x^{10}+64x^{12}.$$

184. Simplify

(1) $\dfrac{1}{x+\sqrt{x^2-1}}+\dfrac{1}{x-\sqrt{x^2-1}}.$

(2) $\sqrt[4]{16}+\sqrt[3]{81}-\sqrt[3]{-512}+\sqrt[3]{192}-7\sqrt[6]{9}.$

185. Solve (1) $\dfrac{5}{6-\dfrac{5}{6-\dfrac{5}{6-x}}}=x.$

(2) $\left.\begin{array}{r}x^2y^2+192=28xy\\ x+y=8\end{array}\right\}.$

186. Simplify

$$\frac{b-c}{a^2-(b-c)^2}+\frac{c-a}{b^2-(c-a)^2}+\frac{a-b}{c^2-(a-b)^2}.$$

187. Solve (1) $x-15\frac{3}{4}+\dfrac{5}{x-15\frac{3}{4}}=6.$

(2) $2(x+y^{-1})=3(x^{-1}-y)=4.$

188. If $xy=ab(a+b)$ and $x^2-xy+y^2=a^3+b^3$ prove that

$$\left(\frac{x}{a}-\frac{y}{b}\right)\left(\frac{x}{b}-\frac{y}{a}\right)=0.$$

189. Find the H.C.F. of

$$(2a^2-3a-2)x^2+(a^2+7a+2)x-a^2-2a$$

and $$(4a^2+4a+1)x^2-(4a^2+2a)x+a^2.$$

190. Multiply $\sqrt{2x}+\sqrt{2(2x-1)}-\dfrac{1}{\sqrt{2x}}$

by $\dfrac{1}{\sqrt{2x}}+\sqrt{2(2x-1)}-\sqrt{2x}.$

191. Divide $a^4b^2+b^4c^2+c^4a^2-a^2b^4-b^2c^4-c^2a^4$

by $a^2b+b^2c+c^2a-ab^2-bc^2-ca^2.$

192. Simplify

(1) $\dfrac{7}{2(x+1)}-\dfrac{1}{6(x-1)}-\dfrac{10x-1}{3(x^2+x+1)}.$

(2) $\left\{\dfrac{\sqrt{x+a}}{\sqrt{x-a}}-\dfrac{\sqrt{x-a}}{\sqrt{x+a}}\right\}\times\dfrac{\sqrt{x^3-a^3}}{\sqrt{(x+a)^2-ax}}.$

193. If p be the difference between any quantity and its reciprocal, q the difference between the square of the same quantity and the square of its reciprocal, shew that

$$p^2(p^2+4)=q^2.$$

194. A man started for a walk when the hands of his watch were coincident between three and four o'clock. When he finished, the hands were again coincident between five and six o'clock. What was the time when he started, and how long did he walk?

195. If n be an integer, shew that $7^{2n+1}+1$ is always divisible by 8.

196. Simplify
$$\frac{\left(p+\frac{1}{q}\right)^p\left(p-\frac{1}{q}\right)^q}{\left(q+\frac{1}{p}\right)^p\left(q-\frac{1}{p}\right)^q}.$$

197. Find the value of

(1) $$\frac{7+3\sqrt{5}}{7-3\sqrt{5}}+\frac{7-3\sqrt{5}}{7+3\sqrt{5}}.$$

(2) $$\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}} \text{ when } x=\frac{2b}{b^2+1}.$$

198. If $a+b+c+d=2s$, prove that

$$4(ab+cd)^2-(a^2+b^2-c^2-d^2)^2=16(s-a)(s-b)(s-c)(s-d).$$

199. A man buys a number of articles for \$5, and sells for \$5.40 all but two at 5 cents apiece more than they cost: how many did he buy?

200. Find the square root of

$$2(81x^4+y^4)-2(9x^2+y^2)(3x-y)^2+(3x-y)^4.$$

201. If $x:a::y:b::z:c$, prove that

$$(bc+ca+ab)^2(x^2+y^2+z^2)=(bz+cx+ay)^2(a^2+b^2+c^2).$$

ANSWERS.

I. a. Page 4.

1. 70.	**2.** 125.	**3.** 105.	**4.** 343.	**5.** 30.
6. 32.	**7.** 12.	**8.** 70.	**9.** 6.	**10.** 108.
11. 7.	**12.** 144.	**13.** 48.	**14.** 189.	**15.** 200.
16. 27.	**17.** 1000.	**18.** 3.	**19.** 1.	**20.** 567.
21. 4.	**22.** 125.	**23.** 81.	**24.** 1.	**25.** 243.
26. 512.	**27.** 5.	**28.** 4096.	**29.** 64.	**30.** 90.
31. 24.	**32.** 2.	**33.** 81.	**34.** 64.	**35.** 8.
36. 2401.	**37.** 56.	**38.** 3.	**39.** 48.	**40.** 16.

I. b. Page 5.

1. 700.	**2.** 686.	**3.** 96.	**4.** 135.	**5.** 15.
6. 60.	**7.** 162.	**8.** 0.	**9.** 0.	**10.** 3000.
11. 98.	**12.** 225.	**13.** 3.	**14.** 1.	**15.** $5\frac{1}{3}$.
16. 2.	**17.** 36.	**18.** 160.	**19.** $1\frac{1}{8}$.	**20.** 40.
21. 0.	**22.** 72.	**23.** 2048.	**24.** 81.	**25.** $11\frac{1}{4}$.
26. $13\frac{1}{2}$.	**27.** $\frac{1}{64}$.	**28.** $3\frac{3}{8}$.		

I. c. Page 6.

1. 4.	**2.** 6.	**3.** 8.	**4.** 36.	**5.** 3.
6. 8.	**7.** 8.	**8.** 0.	**9.** 32.	**10.** 60.
11. 0.	**12.** 16.	**13.** $2\frac{2}{3}$.	**14.** $3\frac{1}{3}$.	**15.** $1\frac{1}{7}$.
16. 0.	**17.** $\frac{2}{3}$.	**18.** $2\frac{2}{3}$.	**19.** 0. **20.** 3.	**21.** 8.

I. d. Page 8.

1. 19.	**2.** 0.	**3.** 7.	**4.** 11.	**5.** 21.
6. 6.	**7.** 18.	**8.** 36.	**9.** 6.	**10.** 14.
11. 85.	**12.** 96.	**13.** 36.	**14.** 0.	**15.** 0.
16. 12.	**17.** 24.	**18.** 43.	**19.** 4.	**20.** 8.
21. 12.	**22.** 0.	**23.** 1.	**24.** 6000.	**25.** 17.
26. 16.	**27.** 18.	**28.** 9.	**29.** $3\frac{1}{2}$.	**30.** 49.
31. $\frac{1}{4}$.	**32.** $\frac{3}{16}$.	**33.** 0.	**34.** $1\frac{5}{6}$.	

I. e. Page 8_A.

1. 20, 2, 2, 12, 30. **2.** 3, 5·25, 8, 11·25, 15. **3.** 6.
4. 4·48, 17·76, 27·52, 40. **6.** 504. **8.** The first by 24.
10. 21, 0. **12.** ·196.

II. a. Page 10.

1. $-\$12$. **2.** 4, -2. **3.** 20. **4.** $-6°$.
5. -3 feet. **6.** 24, -4. **7.** A, C, B with $+4$, 0, -2 points.

II. b. Page 12.

1. $47a$. **2.** $24x$. **3.** $39b$. **4.** $151c$. **5.** $-26x$.
6. $-40b$. **7.** $-17y$. **8.** $-66c$. **9.** $-20b$. **10.** $2x$.
11. 0. **12.** $-16f$. **13.** $-s$. **14.** $7y$. **15.** 0.
16. $2ab$. **17.** x^2. **18.** $-14a^2x$. **19.** $-21a^3$. **20.** $-16x^3$.
21. 0. **22.** $-19x^4$. **23.** $-43abcd$. **24.** $\frac{11}{6}x$. **25.** $\frac{8}{5}a$.
26. $-3b$. **27.** $-x^2$. **28.** $-\frac{5}{6}ab$. **29.** $\frac{5}{4}x$. **30.** $-5x^2$.

III. a. Page 16.

1. 0. **2.** $4a+4b+4c$. **3.** 0.
4. $4x+4y+4z$. **5.** $3a+5b-2c$. **6.** $b-c$.
7. $39a-5b+4c$. **8.** $5c$. **9.** $3ax-3by+3cz$.
10. $22p-18q-20r$. **11.** ab. **12.** $-20ab+ca$.
13. $5ab+bc$. **14.** $pq+qr+rp$. **15.** $6x$.
16. $20a$. **17.** $2xy+2zx$. **18.** $14ab-11bc$.
19. $13z$. **20.** $a+b+c$.

III. b. Page 18.

1. abc. **2.** x^2+xy+y^2. **3.** $a^2+3ab-2b^2$.
4. $yz+zx+xy$. **5.** $3x^2+2xy-y^2$. **6.** $-2x^3+x^2+4x+2$.
7. x^2+7x. **8.** $15x^2-32x-18$. **9.** $15x^3-4x^2+3x-1$.
10. $a^3+b^3+c^3$. **11.** $a^3+b^3+c^3+d^3$. **12.** x^3+x^2+x+3.
13. $9a^3-3a^2$. **14.** $3x^2-2y^2-2xy-4yz-3xz$.
15. $-x^3+x^2+2y^2+y$. **16.** $3x^2y+xy^2$. **17.** $2a^2b$.
18. $x^5-x^4y-y^5$. **19.** $a^3+b^3+c^3-3abc$.

20. $x^3+x^2y+7xy^2+3y^3$. 21. $\frac{1}{4}a-\frac{2}{3}b$.

22. $-3a-\frac{1}{2}b$. 23. $-\frac{7}{3}a+\frac{2}{3}b-\frac{1}{2}c$. 24. $\frac{3}{8}a-\frac{4}{5}b-\frac{15}{4}c$.

25. $\frac{1}{3}x^2-\frac{4}{3}xy+\frac{1}{2}y^2$. 26. $\frac{5}{6}a^2+\frac{3}{5}ab-\frac{1}{6}b^2$.

27. $\frac{3}{8}x^2-\frac{2}{5}xy-\frac{1}{2}y^2$. 28. $-\frac{1}{4}x^3+\frac{3}{8}ax^2+\frac{5}{8}a^2x$.

29. $-\frac{1}{4}x^2-xy+\frac{3}{5}y^2$. 30. $-a^3-\frac{1}{2}a^2b+\frac{1}{4}ab^2+b^3$.

IV. a. Page 21.

1. $-2a-2c$. 2. $3a-5b-4c$. 3. $13x+18y-19z$.
4. $-5a+30b-4c$. 5. $11x+13y-16z$. 6. $12ab-10bc-10cd$.
7. $21a-13b-33c$. 8. $11x+26y+22z$. 9. $2ac+2bd$.
10. $2ab-2cd+2ac-2bd$. 11. $-cd-ac-bd$.
12. $2xy$. 13. $-3x^3-x^2-2x+1$.
14. $-12x^2y+21xy^2+15xyz$. 15. $\frac{1}{6}a-\frac{3}{2}b+\frac{5}{6}c$.
16. $\frac{1}{4}x+\frac{3}{2}y-\frac{2}{3}z$. 17. $-\frac{5}{2}a-\frac{10}{3}b+\frac{1}{2}c$.
18. $x-y+\frac{1}{5}z$. 19. $-\frac{4}{3}x-\frac{4}{3}z$. 20. $-\frac{5}{6}x+\frac{13}{6}y$.

IV. b. Page 22.

1. $7xy-7yz+18xz$. 2. $-12x^2y^2+8x^3y+21xy^3$.
3. $-12+9ab+6a^2b^2$. 4. $-2a^2bc+6b^2ca+5c^2ab$.
5. $-12a^2b+15ab^2-5cd$. 6. $-16x^2y+10xy^2-2x^2y^2$.
7. $20a^2b^2+16a^2b$. 8. $9x^2-9x+9$.
9. x^3+3x^2+5x+7. 10. $-17a^2x^2+13x^2+20$.
11. $2x^2-2x$. 12. $6x^2y+2y^3$. 13. a^3-c^3-abc.
14. $3x^3+10x^2y-10xy^2$. 15. $4x^4-5x^3-2x^2-x+2$.
16. $-4a^3+4b^3-2c^3+10abc$. 17. $-x^5+2x^4+x^3-x^2+2x-2$.
18. $4a^5-7a^4-5a^3+9a^2-a-7$. 19. $-5a^2b-14ab^2+a^3b^3+b^4$.
20. $-a^3+22a^2b-16ab^2+2b^3$. 21. $2x^2-\frac{4}{3}xy-\frac{1}{2}y^2$.
22. $\frac{4}{3}a^2-\frac{7}{2}a-\frac{1}{2}$. 23. $-\frac{1}{6}x^2-\frac{5}{6}x+\frac{7}{6}$. 24. $\frac{5}{8}x^2+\frac{1}{6}ax-\frac{1}{3}$.
25. $\frac{3}{4}x^3-\frac{1}{2}x^2y-\frac{1}{6}y^2$. 26. $-\frac{1}{8}a^3-\frac{2}{3}a^2x-\frac{1}{2}ax^2$.

Miscellaneous Examples I. Page 23.

1. (1) $2x+x^2$; (2) $-3a+b$. **2.** $2a+2c$.
3. (1) 21; (2) 108. **4.** (1) 11; (2) 18. **5.** $7x^3-10x^2$.
6. $8a^3-2a$. **9.** $2x^3-2x^2$. **11.** $2a-(3b+5c)$.
12. 47; 12. **13.** $-y^2+y$. **15.** 36. **16.** 0. **17.** a^2b.
18. $x+2z$. **20.** $7xy$. **21.** 8. **23.** $4a$. **24.** 118.
25. 30 B.C. **26.** $2x^3-2x$. **28.** $a+b-(c-d)$.
29. $a+3b$ miles south of O. **30.** $2x^2+7x-3$.

V. a. Page 27.

1. $35x^7$. **2.** $20a^{11}$. **3.** $56a^4b^3$. **4.** $30x^4y^2$.
5. $8a^3b^6$. **6.** $6a^2bc^4$. **7.** $4a^6b^6$. **8.** $10a^3b$.
9. $28a^7b^3$. **10.** $5a^4b^3x^2y^2$. **11.** $6a^2x^7y^3$. **12.** $abcxyz$.
13. $15a^7b^8x^4$. **14.** $28a^3b^3x^5$. **15.** $40a^2cx^2$. **16.** $30a^3x^6y^3$.
17. $2x^7y^8$. **18.** $3a^5x^9y^{16}$. **19.** $a^4b^2+a^3b^2c$.
20. $20a^3b^2x^3-28a^2b^2x^4$. **21.** $10x^3+6x^2y$.
22. $a^5b+a^3b^3-a^3bc^2$. **23.** $ab^2c^2+a^2bc^2-a^2b^2c$.
24. $20a^4bc^3+12a^2b^3c^3-8a^2bc^5$. **25.** $15x^5y+3x^4y^2-21x^5y^2$.
26. $48x^5y^3-40x^4y^4+56x^3y^5$. **27.** $6a^5b^3c-7a^3b^4c^2$.

V. b. Page 30.

1. 36. **2.** -48. **3.** 5. **4.** 24. **5.** -16.
6. -12. **7.** -9. **8.** -24. **9.** -168. **10.** 480.
11. -16. **12.** 375. **13.** 500. **14.** 140. **15.** -2000.
16. 500. **17.** -180. **18.** -56. **19.** -1000. **20.** -224.
21. 40. **22.** -63. **23.** 118. **24.** -130. **25.** -54.
26. 3. **27.** 1. **28.** 0. **29.** 29. **30.** -13.

V. c. Page 31.

1. $-3a^2x^2$. **2.** $14a^2b^2x^2$. **3.** $-a^3b^3$. **4.** $-60x^3y^2$.
5. $3a^3b^4c^5d^6$. **6.** $-5x^3y^4z^2$. **7.** $-36x^2y^2z-48xy^2z^2$.
8. $a^3b^2c^6-a^2b^2c^4$. **9.** $3x^2+3xy+3xz$. **10.** $a^3bc-ab^3c+abc^3$.
11. $a^2b^2c-ab^2c^2+a^2bc^2$. **12.** $14a^4b^3+28a^3b^4$.
13. $15x^3y^2-18x^2y^3+24x^3y^3$. **14.** $56x^6y^4+40x^4y^6$.
15. $-5x^2y^3z^2+3x^2y^2z^3-8x^3y^2z^2$. **16.** $-48x^5y^3z^5+96x^4y^2z^4$.
17. $91x^4y^5+105x^5y^4$. **18.** $-8x^2y^2z^2+10x^4y^2z^4$.
19. $-a^2b^2c^2+a^3b^2c^2+a^2b^3c^2$. **20.** $a^3b^2c-a^2b^3c+a^2b^2c^2$.

21. $-3a^2+\frac{9}{2}ab-6ac.$ 22. $-\frac{5}{2}x^2+\frac{5}{3}xy+\frac{10}{3}x.$

23. $\frac{1}{4}a^2x-\frac{1}{16}abx-\frac{3}{8}acx.$ 24. $-2a^5x^3+\frac{7}{2}a^4x^4.$

25. $\frac{5}{2}a^4x^2-\frac{5}{3}a^3x^3+a^2x^4.$ 26. $\frac{21}{2}x^3y-x^2y^2.$

27. $\frac{1}{2}x^5y^2-3x^3y^4.$ 28. $-x^8y^3+\frac{16}{49}x^5y^6.$

V. d. Page 32.

1. $x^2+15x+50.$ 2. $x^2-25.$ 3. $x^2-17x+70.$
4. $x^2+3x-70.$ 5. $x^2-3x-70.$ 6. $x^2+17x+70.$
7. $x^2-36.$ 8. $x^2+4x-32.$ 9. $x^2-13x+12.$
10. $x^2+11x-12.$ 11. $x^2-225.$ 12. $-x^2+18x-45.$
13. $x^2+5x+6.$ 14. $-x^2+14x-49.$ 15. $x^2-25.$
16. $x^2+x-182.$ 17. $x^2+x-306.$ 18. $x^2-x-380.$
19. $x^2-256.$ 20. $-x^2+42x-441.$ 21. $2x^2+13x-24.$
22. $2x^2-13x-24.$ 23. $2x^2-11x+5.$ 24. $2x^2-7x+5.$
25. $6x^2+11x-35.$ 26. $6x^2-11x-35.$ 27. $10x^2+3x-18.$
28. $10x^2-3x-18.$ 29. $9x^2-25y^2.$ 30. $9x^2-30xy+25y^2.$
31. $a^2+ab-6b^2.$ 32. $a^2+ab-56b^2.$ 33. $3a^2-30ab+48b^2.$
34. $a^2-4ab-45b^2.$ 35. $x^2+ax-bx-ab.$ 36. $x^2-ax+bx-ab.$
37. $x^2-2ax+3bx-6ab.$ 38. $a^2x^2-b^2y^2.$
39. $x^2y^2-a^2b^2.$ 40. $4p^2q^2-9r^2.$

V. e. Page 34.

1. $a^2+2ab+b^2-c^2.$ 2. $a^2-4b^2+4bc-c^2.$
3. $a^4+a^2b^2+b^4.$ 4. $x^3+4x^2y+3xy^2+12y^3.$
5. $x^4-4x^2+8x+16.$ 6. x^6+y^6 7. $x^3-y^3.$
8. $a^4+4a^2x^2+16x^4.$ 9. $64a^3-27b^3.$ 10. $x^4-a^4.$
11. $x^4+2x^3-7x^2-8x+12.$ 12. $4x^5-x^3+4x.$
13. $a^6+a^3b^3.$ 14. $x^5-2x^4-4x^3+19x^2-31x+15.$
15. $a^5+4ab^4.$ 16. $8x^3-27y^3.$
17. $-x^4+4x^3y-x^2y^2-4xy^3-y^4.$ 18. $a^6-a^4b^4+2a^3b^3+b^6.$
19. $x^4-2x^2y^2+y^4.$ 20. $a^2b^2+c^2d^2-a^2c^2-b^2d^2.$
21. $75a^5b^3-28a^3b^5+13a^2b^6-12ab^7.$ 22. $81x^4-256a^4.$
23. $a^4-25a^2b^2-10ab^3-b^4.$ 24. $x^3+3xy+y^3-1.$
25. $a^3+b^3+c^3-3abc.$ 26. $x^5+y^5.$ 27. $x^{15}+y^{10}.$

28. a^6-2a^3+1. **29.** $a^2x^3+27a^2y^6$. **30.** $x^6+2x^3y^3+y^6$.

31. $\frac{1}{4}a^3+\frac{1}{72}a-\frac{1}{12}$. **32.** $\frac{1}{4}x^3-\frac{5}{6}x^2+\frac{1}{12}x+\frac{1}{2}$.

33. $\frac{2}{9}x^3-\frac{3}{4}y^3$. **34.** $\frac{9}{8}x^4-\frac{3}{2}ax^3+\frac{1}{2}a^2x^2-\frac{2}{9}a^4$.

35. $\frac{1}{4}x^4-\frac{43}{36}x^2+\frac{9}{16}$. **36.** $\frac{1}{4}a^4+x^4$.

V. f. Page 36.

1. $x^2+3x-40$. **2.** x^2+5x-6. **3.** $x^2+7x-30$.
4. x^2+4x-5. **5.** $x^2-2x-63$. **6.** $x^2-18x+80$.
7. $x^2+7x-44$. **8.** x^2+2x-8. **9.** x^2-4.
10. a^2-1. **11.** $a^2+4a-45$. **12.** $a^2+9a-36$.
13. $a^2-4a-32$. **14.** a^2-64. **15.** $a^2+7a-78$.
16. a^2+6a+9. **17.** a^2-121. **18.** $a^2-16a+64$.
19. $x^2-ax-6a^2$. **20.** $x^2+ax-30a^2$. **21.** x^2-9a^2.
22. $x^2+2xy-8y^2$. **23.** x^2-49y^2. **24.** $x^2-6xy+9y^2$.
25. $a^2+6ab+9b^2$. **26.** $a^2+5ab-50b^2$. **27.** $a^2-17ab+72b^2$.
28. $2x^2-x-10$. **29.** $2x^2-9x+10$. **30.** $2x^2-3x-9$.
31. $3x^2+2x-1$. **32.** $4x^2+8x-5$. **33.** $6x^2+5x-21$.
34. $8x^2+6x-9$. **35.** $9x^2-64$. **36.** $4x^2-20x+25$.
37. $9x^2-3xy-2y^2$. **38.** $9x^2+12xy+4y^2$. **39.** $4x^2+4xy-35y^2$.
40. $25x^2-9a^2$. **41.** $2x^2+5ax-25a^2$. **42.** $4x^2+4ax+a^2$

VI. a. Page 40.

1. $3x$. **2.** $-3x$. **3.** $-5x^3$. **4.** $-bx$.
5. xy^2. **6.** $-a^2$. **7.** $4ac$. **8.** $-4a^2b^4c^5$.
9. a^4c^6. **10.** $3x^3y^5z^2$. **11.** $4x^2$. **12.** $6a^6$.
13. $5a^4$. **14.** $7a^2b^3$. **15.** -1. **16.** $-7ab^2$.
17. $-8b^2x$. **18.** $10y^2$. **19.** $x-2y$. **20.** x^2-3x+1.
21. $x^4-7x^3+4x^2$. **22.** $10x^4-8x^3+3x$.
23. $-3x^2+5x$. **24.** $3x-4$. **25.** $3x^3+4x$. **26.** $2x^2y-3xy^2$.
27. $-a+b+c$. **28.** $a-b-b^2$. **29.** $-x^2+3xy+4y^2$.
30. $-2x^3y^3+4x^2y-3y^2$. **31.** $2a-3b+4c$.
32. $-\frac{1}{3}x^2+2y^2$. **33.** $3x-2y-4$.
34. $-\frac{6}{7}a^2x^2+\frac{3}{2}ax^3$. **35.** $\frac{2}{3}a-\frac{1}{6}b-c$.

VI. b. PAGE 42.

1. $x+2$. 2. $x-4$. 3. $a-6$. 4. $a-24$.
5. $3x+1$. 6. $x+5$. 7. $5x+1$. 8. $x+7$.
9. $5x+1$. 10. $x+11$. 11. $x+5$. 12. $3x+1$.
13. $3x+7$. 14. $3x-7$. 15. $3x-5$. 16. $4x-7$.
17. $4a+3x$. 18. $5a-x$. 19. $3a+4c$. 20. $3a-5c$.
21. $6x+5y$. 22. $8x+3y$. 23. x^2+14x. 24. $4x^2-x$.
25. $9x^2+9x+5$. 26. $2a^2-5a+3$.
27. $3+3a+a^2$. 28. $8-36x+54x^2-27x^3$.

VI. c. PAGE 44.

1. $x-4$. 2. $y+1$. 3. $2m-3$. 4. $2a^2-3a$.
5. x^2-x+1. 6. a^2-3a+1; rem. $a-6$. 7. a^2+3a+2.
8. $2x^2+x-1$; rem. $3x+4$. 9. x^3-2x^2+x+1.
10. x^3-3x^2+2x-1. 11. $10x^2-3x-12$; rem. $7x-45$.
12. $7y^2+5y-3$; rem. $-39y+27$. 13. $2k^2-5k+2$.
14. $5-7m-m^3$. 15. x^2+5x+6.
16. x^2-2x+3; rem. $31x-15$. 17. $12+8x+x^2$.
18. $7x^2+5xy+2y^2$. 19. x^2-xy+y^2; rem. x^2.
20. x^3+x-y. 21. $a^6+a^3b^3+b^6$.
22. $x^7-x^6y+x^4y^3-x^3y^4+xy^6-y^7$.
23. $x^6+2x^5y^2-3x^4y^4-6x^3y^6+2x^2y^8+4xy^{10}+y^{12}$.
24. $a^2+2ab+b^2+a+b+1$. 25. $x^5-x^4y+xy^4-y^5$.
26. $a^{10}+a^8b^2+a^6b^4+a^4b^6+a^2b^8+b^{10}$.
27. $a^8-2a^6b^2+3a^4b^4-2a^2b^6+b^8$. 28. $1+a+a^2+2x-2ax+4x^2$.
29. $\frac{1}{4}a^2-3ax+9x^2$. 30. $\frac{1}{9}a^2-\frac{1}{6}a+\frac{1}{16}$.
31. $\frac{6}{25}a^3-\frac{3}{5}a^2c+\frac{3}{2}ac^2$. 32. $\frac{3}{8}a^2-\frac{1}{4}a-\frac{2}{3}$.
33. $6x-\frac{1}{3}y-\frac{1}{2}$. 34. $\frac{4}{9}a^4+\frac{1}{2}a^3x+\frac{9}{16}a^2x^2+\frac{81}{128}ax^3$.

VII. a. PAGE 47.

1. $a+b-c$. 2. a. 3. $a+3b-4c$. 4. $3a-b-c$.
5. $-2a-4b-2c$. 6. $-a+b-c$. 7. $b-a$.
8. $x-y$. 9. $2a-2b$. 10. $-2x-5y$. 11. $x-a$.
12. $2a-b-d$. 13. $-3c+4y$. 14. $-x+2y+6z$.
15. $-5x$. 16. $-25x+2y$. 17. $11x-36y$.
18. $2x-2z$. 19. $2a$. 20. a.

VII. b. Page 48.

1. $3a.$ **2.** $a.$ **3.** $6a+2b-2c-2d.$

4. $2x-3y+12z.$ **5.** $b.$ **6.** $21a+b.$

7. $2b+4c.$ **8.** $-a^2+8b^2-9c^2.$ **9.** $-2a+6b+2c-2d.$

10. $4a+b+c.$ **11.** $-50c.$ **12.** $-11a-2b.$

13. $-a+b+5c.$ **14.** $-2a+10b-11c.$ **15.** $-227a+216b+84.$

16. $2a-12c+84d.$ **17.** $3a+4x.$ **18.** $-10a.$

19. $4a.$ **20.** $0.$ **21.** $\frac{11}{5}a-2b.$

22. $12x-30y.$ **23.** $a-\frac{13}{3}b+\frac{10}{3}c.$ **24.** $0.$

VII. c. Page 50.

1. $(a+2)x^4+(b-5)x^2+(2b-3)x+5.$

2. $(5a-b)x^3+(3b-4)x^2+(c-2)x+ab-7.$

3. $(9a-7)x^3+(5a-3)x^2+(7-2c)x+2.$

4. $(2c-a^2)x^5+(1-3b)x^4+(4d-3ab)x.$

5. $-(a^2+b)x^4-(2b-5)x^3-(3-a)x^2.$

6. $-(ab-7)x^5-(abc-7)x^3-(3c^2-5a)x.$

7. $-(c-a^2)x^3-(b+5-a)x^2.$

8. $-(a+c+7-3b^2)x^4-(b+5c^2)x.$

9. $(a-b)x^3-(b+2c)x^2-(b+c+d)x.$

10. $(5a+4c)x^3+(3a-6b+7c)x^2+(2a-7b)x.$

11. $(3a+2c)x^3+(a+8b)x^2-(8a+9b)x.$

12. $(6b+1)x^5-(a+2b)x^4-(2a+3c)x.$

13. $(a+b)x^3-(a+b)x^2+(a-b)x.$

VII. d. Page 51.

1. $(a-c)x^3+(b+c)x^2-(2c+1)x.$

2. $(1-b)x^3+(a+1)x^2+(b-1)x-1.$

3. $(a^2-5a+2)x^3+(2a-b)x^2-(a+5)x.$

4. $(a-p+1)x^2+(b+q+2)x-c-r+3.$

5. $(p+q-1)x^3+(p+q)x^2-(p+q)x+q.$

6. $acx^3+(2a+bc)x^2+(2b+c)x+2.$

7. $acx^3-(2a+bc)x^2+(3a+2b)x-3b.$

8. $apx^3+(aq-bp)x^2-(bq+cp)x-cq.$

9. $2bx^3-(3b-2c)x^2-(b+3c)x-c.$

10. $ax^3-(a+2b)x^2+(2b+3c)x-3c.$

11. $apx^3-(2a+3p)x^2+(6-aq)x+3q.$

12. $x^6-(a^2+2b)x^4+(2ac+b^2)x^2-c^2.$

13. $a^2x^6+(6a-1)x^4+(9-2b)x^2-b^2.$

14. $x^8-(a^2+2b)x^6+(2ac+b^2+2d)x^4-(2bd+c^2)x^2+d^2.$

VIII. (1). Page 54_A.

1.	3.	**2.**	5.	**3.**	4.	**4.**	6.	**5.**	2.	**6.**	5.
7.	2.	**8.**	3.	**9.**	5.	**10.**	7.	**11.**	$3\frac{1}{2}$.	**12.**	2.
13.	2.	**14.**	4.	**15.**	3.	**16.**	5.	**17.**	$\frac{1}{2}$.	**18.**	$2\frac{1}{2}$.
19.	1.	**20.**	$1\frac{1}{2}$.	**21.**	2.	**22.**	1.	**23.**	$2\frac{1}{2}$.	**24.**	$6\frac{2}{3}$.
25.	$\frac{5}{8}$.	**26.**	$\frac{7}{12}$.	**27.**	$\frac{8}{21}$.	**28.**	$1\frac{13}{27}$.	**29.**	12.	**30.**	15.
31.	$4\frac{1}{2}$.	**32.**	-4.								

VIII. a. Page 55.

1.	5.	**2.**	4.	**3.**	7.	**4.**	4.	**5.**	3.	**6.**	1.
7.	5.	**8.**	3.	**9.**	15.	**10.**	13.	**11.**	13.	**12.**	5.
13.	1.	**14.**	16.	**15.**	10.	**16.**	30.	**17.**	5.	**18.**	1.
19.	2.	**20.**	1.	**21.**	1.	**22.**	2.	**23.**	3.	**24.**	1.
25.	4.	**26.**	3.	**27.**	3.	**28.**	3.	**29.**	1.	**30.**	4.
31.	7.	**32.**	3.	**33.**	4.	**34.**	4.	**35.**	1.	**36.**	1.
37.	2.	**38.**	2.	**39.**	1.	**40.**	2.				

VIII. b. Page 58.

1.	20.	**2.**	15.	**3.**	8.	**4.**	16.	**5.**	25.	**6.**	17.
7.	13.	**8.**	10.	**9.**	7.	**10.**	4.	**11.**	$-\frac{1}{7}$.	**12.**	$\frac{1}{7}$.
13.	5.	**14.**	7.	**15.**	6.	**16.**	10.	**17.**	6.	**18.**	8.
19.	7.	**20.**	25.	**21.**	$3\frac{1}{7}$.	**22.**	8.	**23.**	12.	**24.**	5.
25	5.	**26.**	12.	**27.**	$\frac{4}{7}$.	**28.**	$-5\frac{1}{6}$.	**29.**	8.	**30.**	$66\frac{2}{3}$.
31.	7.	**32.**	7.	**33.**	2.	**34.**	12.	**35.**	27.	**36.**	5.

VIII. c. Page 60.

1. $2\frac{2}{3}$. **2.** 6. **3.** 10. **4.** -6. **5.** $9\frac{2}{3}$. **6.** $1\frac{1}{3}$.
7. -12. **8.** $\frac{3}{8}$. **9.** $1\frac{4}{5}$. **10.** $-\frac{3}{4}$. **11.** $\frac{4}{5}$. **12.** $-\frac{2}{21}$.
13. $1\frac{1}{2}$. **14.** $-\frac{2}{3}$. **15.** $1\frac{3}{4}$. **16.** 12. **17.** $3\frac{6}{7}$. **18.** $2\frac{1}{4}$.
19. $\frac{5}{7}$. **20.** $1\frac{2}{5}$. **21.** $\frac{3}{7}$.

IX. a. Page 62.

1. $y-x$. **2.** $\frac{a}{3}$. **3.** $5b$. **4.** $3d-2c$.
5. $2k$. **6.** $100-x$. **7.** $\frac{b}{a}$. **8.** $20-c$.
9. $\frac{5}{6}a$. **10.** $\frac{10}{x}$. **11.** $x+11$. **12.** $c-20$.
13. $90-x$. **14.** $x-30$. **15.** 20. **16.** $\frac{2x}{5}$.
17. $36-x$. **18.** $x+a$. **19.** $5x$ days. **20.** 4.
21. $\frac{6x}{25}$. **22.** $\frac{x}{4}$. **23.** xy miles. **24.** $\frac{y}{x}$ miles.
25. $\frac{60x}{a}$. **26.** $\frac{118}{x}$ hours. **27.** $5p$. **28.** $\frac{44}{x}$.
29. $100a+25b$. **30.** $40-x$. **31.** $100a+25b-c$.
32. $x-6$. **33.** b. **34.** $25x$. **35.** $2a+4b-c$.
36. $\frac{100}{xy}$. **37.** $y-13x$. **38.** $100-x-y-z$.
39. $100x+10y+z-50$. **40.** $2y+2z-x$.

IX. b. Page 65.

1. $x,\ x+1,\ x+2,\ x+3$. **2.** $y-2,\ y-1,\ y$.
3. $x-2,\ x-1,\ x,\ x+1,\ x+2$. **4.** $2n+2$. **5.** $2x-1$.
6. $6n+3$. **7.** $x-a-b$ miles. **8.** $n(a+b)$. **9.** $x+y+5$.
10. $2x+5$. **11.** $mx+y$. **12.** $6x$. **13.** $\$10bc$.
14. $\$\frac{ax}{20}$. **15.** $\$\frac{a^3}{2}$. **16.** $\$\frac{x^2y^2}{50}$. **17.** $3xy$.
18. $\frac{x^2}{9}$. **19.** $\frac{px}{2}$. **20.** $\frac{abc}{300}$. **21.** $\frac{4yz}{x}$.

22. $ab-\frac{c^2}{9}$. **23.** $\frac{3a}{4x}$. **24.** $\frac{bc}{20}$ hours. **25.** $\frac{22a}{15b}$.

26. $\frac{15xy}{22}$. **27.** $\frac{y}{xz}$ days. **28.** yz. **29.** $\frac{y}{10r}$. **30.** $\frac{100p}{ar}$.

31. $p(p-1)(p-2)=y$. **32.** $6n=x$. **33.** $pq=5(a-b)$.

34. $\frac{x}{y}=m+n+10$. **35.** $a+x+5=2(a+5)$; 35; 24.

36. $p-x=3\left(\frac{q}{100}+x\right)$. **37.** $p-5=7(q-5)$.

IX. c. Page 68_A.

1. (i) 272 sq. ft.; (ii) 16 ft.; (iii) 6 ch. 84 lks.

2. (i) 50 cu. ft.; (ii) $4\frac{1}{2}$ cu. ft.; (iii) 5 ft.

3. (i) 49; (ii) 1 hr. 20 min.; (iii) $37\frac{1}{2}$.

4. (i) 144·9 ft.; (ii) 5 secs.

5. 22 in., 38·5 sq. in.; 11 ft., 9·625 sq. ft.

6. (i) 24·64 sq. in.; (ii) 1 ft. 9 in.

7. (i) $2(x+y)$ ft.; (ii) xy sq. ft.; (iii) $2z(x+y)$ sq. ft.

8. 59 ft. 10 in.; 210 sq. ft.; 718 sq. ft. **9.** 10 ft. 6 in.

10. (i) 22 sq. cm.; (ii) 3·6 sq. in. **11.** 1·5 in. **12.** 27 sq. ft.

13. 328. **14.** 15. **15.** 55. **16.** (i) and (iii)

20. (i) 17; (ii) 24; (iii) 40; (iv) 1·6.

21. (i) ·7854; (ii) 96·6; (iii) 294. **22.** 40. **23.** 12.

24. (i) 9780; (ii) 1; (iii) 12; (iv) $-40·5$.

25. 4, $5\frac{1}{5}$, $6\frac{2}{5}$, $7\frac{3}{5}$, $8\frac{4}{5}$, 10.

X. a. Page 71.

1. 17, 12. **2.** 13, 5. **3.** 75. **4.** 20 miles.

5. 15, 43. **6.** 162. **7.** 1. **8.** 50, 55.

9. 27, 28, 29. **10.** 3, 5. **11.** 15, 5. **12.** \$20.

13. 5. **14.** 60, 61. **15.** 6, 3.

16. A \$100, B \$130, C \$150. **17.** 16 quarters, 30 ten-cent pieces.

18. Silk \$1.50, Linen 30 cts. **19.** 48, 12. **20.** 65, 40.

21. 60, 10. **22.** 20 half-dollars, 5 dollars, 10 ten-cent pieces.

23. 25, 5. **24.** 123 runs, 10 byes, 5 wides.

25. 15 ft., 12 ft. **26.** 18 ft., 10 ft.

X. b. Page 73.

1. 54. 2. 24. 3. 60. 4. 35.
5. 75. 6. 24, 25. 7. 224, 252. 8. 49, 50.
9. 50, 51, 52. 10. \$33. 11. 27.
12. 90 Port, 150 Claret. 13. A \$450, B \$180, C \$140.
14. A \$525, B 600, C \$160. 15. \$50.
16. 12 ft. 18 ft. 17. \$12000. 18. 44.

XI. a. Page 74.

1. $2ab$. 2. x^2y^2. 3. $2xy^2z$. 4. abc.
5. $5ab$. 6. $3xy^2z$. 7. $2a^2b^2c^2$. 8. $7ab^2c^3$.
9. $3x^2yz^2$. 10. $2ax$. 11. $7a$. 12. $17abc$.
13. xy. 14. $8a^2b^2c^2$. 15. $25xy$. 16. bx.
17. $5a^3b^3c^2$. 18. abc.

XI. b. Page 75.

In examples 19–29 the H.C.F. stands first; the L.C.M. second.

1. $2a^2bc$. 2. x^3y^2z. 3. $12x^3y^3z$. 4. $20a^2b^2c^3$.
5. $15a^4b^3c^5$. 6. $24abxy$. 7. abc. 8. $a^2b^2c^2$.
9. $12abc$. 10. $12xyz$. 11. $12x^2y^2z^2$. 12. $42a^2b^3$.
13. $a^2b^2c^2$. 14. $30a^2b^2c^2$. 15. $12x^3y^4$. 16. $56x^4y^5$.
17. $210a^3b^3c^3$. 18. $264a^4b^4c^4$. 19. ac, $12abc$. 20. $2y$, $12xyz$.
21. bc, $9ab^2c$. 22. $13a^2bc$, $39a^3bc^2$. 23. $17xy$, $51x^2yz^2$.
24. $5xy^3z$, $75x^3y^3z^2$. 25. b, $30abc$. 26. $17m^2p^2$, $51m^4n^4p^4$.
27. y^2, $x^3y^5z^4$. 28. $5p^2$, $60m^2p^3q^4$. 29. $36k^2m^2n^4$, $216k^3m^3n^5$.

XII. a. Page 76.

1. $\frac{1}{2b}$. 2. $\frac{a}{4b}$. 3. $\frac{2y}{5x}$. 4. $\frac{1}{5ab}$.
5. $\frac{z^2}{xy}$. 6. $\frac{3a}{5c}$. 7. $\frac{3x^2}{4z^2}$. 8. $\frac{2a^2}{3bc}$.
9. $\frac{4n}{5mp}$. 10. $\frac{5m^2p^2}{6n^4}$. 11. $\frac{c}{a^2b}$. 12. $\frac{3xz}{5y^3}$.
13. $\frac{yz^3}{2x}$. 14. $\frac{a^2c^3}{3b^2}$. 15. $\frac{nq}{mp^3}$. 16. $\frac{2np^3}{3m}$.
17. $\frac{3x^2}{5ay^4}$. 18. $\frac{3}{4abc}$. 19. $\frac{2p^2m^2}{3k}$. 20. $\frac{2xyz}{3}$.

XII. b. Page 77.

1. $\frac{2cd^2}{3b}$. 2. $\frac{a^2}{bc}$. 3. $\frac{9ax^2z^2}{bc}$. 4. $\frac{14b^3}{15c^3y}$.

5. $\frac{3mnz^2}{2x}$. 6. $\frac{9mnp}{4k}$. 7. $\frac{x}{2a^2}$. 8. $\frac{400x}{441y^3}$.

9. $\frac{y^3z^2}{nx^4}$. 10. 3. 11. $\frac{7b}{4a}$. 12. $\frac{d^2}{4a^2c}$. 13. $\frac{7acy}{8bdx}$.

14. $\frac{6x^2yz}{5a}$. 15. $\frac{9b^2cz^2}{4x^3y}$. 16. 8. 17. $\frac{p^2q^2y}{10x^2}$. 18. y^2.

XII. c. Page 78.

1. $\frac{4x,\ y}{2a}$. 2. $\frac{4x^3,\ 3y^2}{3x^2y}$. 3. $\frac{ac,\ 2b^2}{2bc}$

4. $\frac{ad,\ bc,\ 2bd}{bd}$. 5. $\frac{6ac,\ b^2}{3bc}$. 6. $\frac{5m,\ 4p}{20n}$.

7. $\frac{3k,\ 2p}{6x}$. 8. $\frac{2m,\ n}{6x}$. 9. $\frac{a^2,\ b^2}{abc}$. 10. $\frac{ax,\ b}{x^2}$.

11. $\frac{2y,\ 3x}{xy}$. 12. $\frac{x^2,\ y^2,\ 3x^2y}{xy}$. 13. $\frac{4x^2,\ 9y^2}{6xy}$.

14. $\frac{8ac,\ 3ab}{10bc}$. 15. $\frac{9ac,\ 5b^2}{21bc}$. 16. $\frac{18,\ 3ab,\ a^2}{9a}$.

XII. d. Page 79.

1. $\frac{5x}{6}$. 2. $\frac{y}{20}$. 3. $\frac{a}{12}$. 4. $\frac{2x^2-15}{3x}$.

5. $\frac{5x+2y}{10}$. 6. $\frac{3a-2b}{12}$. 7. $\frac{3m-2n}{24}$. 8. $\frac{2m-3n}{15}$.

9. $\frac{3x-y}{21}$. 10. $\frac{3a+b}{39}$. 11. $\frac{3p-q}{48}$. 12. $\frac{15m-n}{36}$.

13. $\frac{22x}{15}$. 14. $\frac{9x}{20}$. 15. $\frac{x}{4}$. 16. $\frac{6a-4b}{15}$.

17. $\frac{11a}{30}$. 18. $\frac{5x}{24}$. 19. $\frac{7x}{18}$. 20. $\frac{5x}{4}$.

21. $\frac{31x}{36}$. 22. $\frac{17x}{24}$. 23. $\frac{bx-ay}{ab}$. 24. $\frac{9bx+2ay}{3ab}$.

25. $\frac{ac+b}{c}$. 26. $\frac{xz-y}{z}$. 27. $\frac{a^2-3b^2}{3a}$. 28. $\frac{a^3+b^3}{a}$.

29. $\frac{x^3-2y^2}{2x^2}$. 30. $\frac{p^5-k^5}{p^2}$.

Miscellaneous Examples II. Page 80.

1. $3x^2+7x-8$. **2.** $13z$. **3.** 20.

4. $a^2+b^2+c^2$. **5.** $x^5-11x-10$. **6.** (1) $\frac{1}{2}$; (2) -3.

7. x^2+2x-3. **8.** $-4a+5b$. **9.** $5x$.

10. $4x^2-6x-1$. **11.** (1) $x^2+14x-51$; (2) $24x^2-55x-24$.

12. (1) $\frac{1}{4}$; (2) 1. **13.** $-ab$. **14.** $\frac{8}{5}$.

15. $16a^2+2ab$. **16.** $x^5+4x^4+48x-32$.

17. $29a$. **18.** (1) -2; (2) 41. **19.** $3p^3-5p^2+2p$.

20. $6a+2c-2d$. **21.** $2x^3-x^2-x$. **22.** 1.

23. 1935. **24.** 4. **25.** $4m-5n$.

26. $3x-9$. **27.** 0. **28.** (1) -15; (2) 4.

29. $3y^3-9y^2+2y-1$. **30.** A \$800, B \$320.

31. 14. **32.** $6m^4-96$. **33.** $x-2$.

34. $ap+bq$ miles; $\frac{ap+bq}{c}$ hours. Numerically, 55 miles; 5 hours.

35. (1) $\frac{1}{7}$; (2) $7\frac{1}{13}$. **36.** \$1080, \$360, \$180.

XIII. a. Page 86.

1. $x=2$, $y=1$. **2.** $x=3$, $y=5$. **3.** $x=2$, $y=3$.

4. $x=4$, $y=-1$. **5.** $x=1$, $y=2$. **6.** $x=3$, $y=4$.

7. $x=5$, $y=6$. **8.** $x=1$, $y=2$. **9.** $x=3$, $y=1$.

10. $x=2$, $y=1$. **11.** $x=1$, $y=3$. **12.** $x=1$, $y=1$.

13. $x=7$, $y=5$. **14.** $x=10$, $y=3$. **15.** $x=5$, $y=12$.

16. $x=7$, $y=8$. **17.** $x=6$, $y=8$. **18.** $x=5$, $y=8$.

19. $x=-7$, $y=-3$. **20.** $x=17$, $y=-19$. **21.** $x=1$, $y=2$.

XIII. b. Page 87.

1. $x=12$, $y=8$. **2.** $x=10$, $y=6$. **3.** $x=18$, $y=12$.

4. $x=20$, $y=15$. **5.** $x=45$, $y=35$. **6.** $x=51$, $y=17$.

7. $x=20$, $y=60$. **8.** $x=14$, $y=15$. **9.** $x=-2$, $y=4$.

10. $x=3$, $y=5$. **11.** $x=7$, $y=3$. **12.** $x=5$, $y=4$.

13. $x=3$, $y=-4$. **14.** $x=19$, $y=3$. **15.** $x=12$, $y=-4$.

16. $x=13$, $y=7$.

XIII. c. Page 90.

1. $x=1,\ y=2,\ z=3$.	**2.** $x=-2,\ y=4,\ z=1$.
3. $x=2,\ y=3,\ z=1$.	**4.** $x=1,\ y=2,\ z=3$.
5. $x=9,\ y=2,\ z=-4$.	**6.** $x=3,\ y=2,\ z=1$.
7. $x=5,\ y=6,\ z=7$.	**8.** $x=1,\ y=2,\ z=3$.
9. $x=2,\ y=-2,\ z=5$.	**10.** $x=4,\ y=-3,\ z=2$.
11. $x=8,\ y=10,\ z=14$.	**12.** $x=3,\ y=9,\ z=15$.
13. $x=6,\ y=8,\ z=5$.	**14.** $x=\frac{3}{2},\ y=\frac{2}{3},\ z=\frac{5}{6}$.
15. $x=6,\ y=2,\ z=1$.	**16.** $x=35,\ y=30,\ z=25$.

XIII. d. Page 92.

1. $x=5,\ y=3$.	**2.** $x=2,\ y=7$.	**3.** $x=3,\ y=2$.
4. $x=\frac{1}{3},\ y=\frac{1}{4}$.	**5.** $x=7,\ y=6$.	**6.** $x=\frac{1}{3},\ y=\frac{1}{5}$.
7. $x=2,\ y=-3$.	**8.** $x=-5,\ y=4$.	**9.** $x=\frac{2}{3},\ y=\frac{3}{4}$.
10. $x=9,\ y=25$.	**11.** $x=\frac{1}{4},\ y=\frac{1}{3}$.	**12.** $x=\frac{1}{5},\ y=\frac{1}{6}$.

13. $x=\frac{1}{2},\ y=\frac{1}{3},\ z=\frac{1}{4}$. **14.** $x=\frac{1}{8},\ y=\frac{1}{12},\ z=\frac{1}{16}$.

15. $x=3,\ y=-2,\ z=1$.

XIV. Page 95.

1. 22, 12. **2.** 55, 18. **3.** 25, 17. **4.** 53, 23.
5. 23, 17. **6.** Tea 85*c*., Sugar 5*c*. **7.** Horse \$23, Cow \$16.
8. *A* 140, *B* \$60, *C* \$70, *D* \$20. **9.** *A* \$99, *B* \$115, *C* \$33, *D* \$23.
10. *A* 36 years, *B* 14 years. **11.** *A* 55 years, *B* 21 years.
12. *A* 5 miles, *B* 4 miles. **13.** *C* $3\frac{1}{2}$ miles, *D* $4\frac{1}{4}$ miles.
14. $\frac{13}{25}$. **15.** $\frac{15}{26}$. **16.** $\frac{2}{15}$. **17.** $\frac{3}{14}$. **18.** 28, 82.
19. 85, 58. **20.** 27. **21.** 72. **22.** \$5, 8 ten-cts.
23. 8 white, 12 black. **24.** 860. **25.** Man \$1.75, Boy 60*c*.
26. 20 lbs., 40 lbs. **27.** 15 miles. **28.** 8 hours.
29. 6 miles, 3 miles an hour. **30.** 45.
31. \$500. **32.** 3 miles, $4\frac{2}{7}$ miles an hour.

XV. a. PAGE 99.

1. $9a^2b^6$. **2.** a^6c^2. **3.** $49a^2b^4$. **4.** $121b^4c^6$.

5. $16a^8b^{10}x^4$. **6.** $25x^4y^{10}$. **7.** $4a^2b^2c^4$. **8.** $9c^2x^6$.

9. $16x^2y^2z^6$. **10.** $\frac{4}{9}a^4b^6$. **11.** $\frac{4x^4}{9y^6}$. **12.** $\frac{16}{9x^4y^2}$.

13. $\frac{49a^2b^2}{9}$. **14.** $\frac{9a^4b^6}{16c^{10}x^8}$. **15.** $\frac{1}{4x^2y^2}$. **16.** $4x^2y^4$.

17. $\frac{25a^2b^6}{4x^2y^2}$. **18.** $169c^{10}x^6$. **19.** $\frac{1}{16a^8}$. **20.** $\frac{9a^{10}}{25x^6}$.

21. $8a^3b^6$. **22.** $27x^9$. **23.** $64x^{12}$. **24.** $-27a^9b^3$.

25. $-125a^3b^6$. **26.** $-b^9c^6x^3$. **27.** $-216a^{18}$. **28.** $-8a^{21}c^6$.

29. $\frac{1}{27y^6}$. **30.** $-\frac{27x^{15}}{125a^9}$. **31.** $343x^9y^{12}$. **32.** $-\frac{8}{27}a^{15}$.

33. $81a^8b^{12}$. **34.** $a^{12}x^6$. **35.** $-32x^{15}y^5$. **36.** $\frac{1}{128a^{14}}$.

37. $\frac{243x^{20}}{32y^{15}}$. **38.** $\frac{256x^{24}}{6561y^8}$. **39.** $-\frac{x^{21}}{2187}$. **40.** $\frac{64x^{30}}{729a^{24}}$.

XV. b. PAGE 101.

1. $a^2+6ab+9b^2$. **2.** $a^2-6ab+9b^2$. **3.** $x^2-10xy+25y^2$.

4. $4x^2+12xy+9y^2$. **5.** $9x^2-6xy+y^2$. **6.** $9x^2+30xy+25y^2$.

7. $81x^2-36xy+4y^2$. **8.** $25a^2b^2-10abc+c^2$. **9.** $p^2q^2-2pqr+r^2$.

10. $x^2-2abcx+a^2b^2c^2$. **11.** $a^2x^2+4abxy+4b^2y^2$.

12. x^4-2x^2+1. **13.** $a^2+b^2+c^2-2ab-2ac+2bc$.

14. $a^2+b^2+c^2+2ab-2ac-2bc$. **15.** $a^2+4b^2+c^2+4ab+2ac+4bc$.

16. $4a^2+9b^2+16c^2-12ab+16ac-24bc$.

17. $x^4+y^4+z^4-2x^2y^2-2x^2z^2+2y^2z^2$.

18. $x^2y^2+y^2z^2+z^2x^2+2xy^2z+2x^2yz+2xyz^2$.

19. $9p^2+4q^2+16r^2-12pq+24pr-16qr$.

20. $x^4-2x^3+3x^2-2x+1$. **21.** $4x^4+12x^3+5x^2-6x+1$.

22. $x^2+y^2+a^2+b^2-2xy+2ax-2bx-2ay+2by-2ab$.

23. $4x^2+9y^2+a^2+4b^2+12xy+4ax-8bx+6ay-12by-4ab$.

24. $m^2+n^2+p^2+q^2-2mn-2mp-2mq+2np+2nq+2pq$.

25. $\frac{a^2}{4}+4b^2+\frac{c^2}{16}-2ab+\frac{ac}{4}-bc$. **26.** $\frac{a^2}{9}+9b^2+\frac{9}{4}-2ab-a+9b$.

27. $\frac{4x^4}{9}-\frac{4x^3}{3}+3x^2-3x+\frac{9}{4}$.

XV. c. Page 101.

1. $x^3+3ax^2+3a^2x+a^3$.
2. $x^3-3ax^2+3a^2x-a^3$.
3. $x^3-6x^2y+12xy^2-8y^3$.
4. $8x^3+12x^2y+6xy^2+y^3$.
5. $27x^3-135x^2y+225xy^2-125y^3$.
6. $a^3b^3+3a^2b^2c+3abc^2+c^3$.
7. $8a^3b^3-36a^2b^2c+54abc^2-27c^3$.
8. $125a^3-75a^2bc+15ab^2c^2-b^3c^3$.
9. $x^6+12x^4y^2+48x^2y^4+64y^6$.
10. $64x^6-240x^4y^2+300x^2y^4-125y^6$.
11. $8a^9-36a^6b^2+54a^3b^4-27b^6$.
12. $125x^{15}-300x^{10}y^4+240x^5y^8-64y^{12}$.
13. $a^3-2a^2b+\frac{4}{3}ab^2-\frac{8}{27}b^3$.
14. $\frac{1}{27}a^3+\frac{2}{3}a^2+4a+8$.
15. $\frac{1}{27}x^6-x^5+9x^4-27x^3$.
16. $\frac{1}{216}a^3+\frac{1}{6}a^2x+2ax^2+8x^3$.

XVI. a. Page 103.

1. $2ab^2$.
2. $3x^3y$.
3. $5x^2y^3$.
4. $4a^2bc^3$.
5. $9a^3b^4$.
6. $10x^4$.
7. $a^{10}b^3c^2$.
8. a^4bc^6.
9. $8x^6y^9$.
10. $\frac{6}{a^{18}}$.
11. $\frac{a^8b^4}{4}$.
12. $\frac{17y^2}{5}$.
13. $\frac{18x^6}{13y^3}$.
14. $\frac{9a^9}{6b^6}$.
15. $\frac{16xy^2}{17p^7}$.
16. $\frac{20a^{20}b^{10}}{9x^5y^9}$.
17. $3a^2bc$.
18. $-2a^4b^3$.
19. $4x^2yz^4$.
20. $-7a^4b^6$.
21. $-\frac{x^4y^3}{5}$.
22. $\frac{2x^3}{9y^5}$.
23. $\frac{5ab^2}{6x^2y^3}$.
24. $-\frac{3x^9}{4y^{21}}$.
25. a^2x^3.
26. x^2y^3.
27. $2xy^2$.
28. $3a^3b$.
29. $2ax^8$.
30. $-x^2y^3$.
31. $\frac{2}{a^9b^8}$.
32. $\frac{a^3x^5}{b^{10}}$.
33. $\frac{a^2}{b^3c^4}$.

XVI. b. Page 106.

1. $x+2y$.
2. $3a+2b$.
3. $x-5y$.
4. $2x-3y$.
5. $9x+y$.
6. $5x-3y$.
7. x^2-y^2.
8. $1-a^3$.
9. a^2-a+1.
10. $2x^2-3x+5$.
11. $3x^2-2x-1$.
12. x^2-2x+1.
13. $2a^2+a-2$.
14. $1-5x+x^2$.
15. $2x+3y-5z$.
16. $4x^3+2x^4-x^5$.
17. $x^3-11x+17$.
18. $5x^2-3ax+4a^2$.
19. $2x^2+y^2-3z^2$.
20. $ab-2ac+3bc$.
21. $2a^2+b^2-3c^2$.
22. $2x^2-xy+3y^2$.
23. $3x^2-5x+7$.
24. $1-2x+3x^2-4x^3$.
25. ax^5-2bx^2+3c.

XVI. c. Page 107.

1. $\frac{x}{2}-3.$ 2. $2-\frac{x}{y}.$ 3. $\frac{x}{5}+y.$ 4. $\frac{x}{y}+5.$ 5. $\frac{x}{2y}-2.$

6. $\frac{x}{y}-\frac{a}{b}.$ 7. $\frac{8x}{3y}+2.$ 8. $\frac{3x}{5}-\frac{5}{3x}.$ 9. $\frac{a^2}{8}+\frac{a}{2}-1.$

10. $x^2+x-\frac{1}{2}.$ 11. $a^2-\frac{3}{2}a+\frac{5}{3}.$ 12. $x^2-3x+\frac{1}{3}.$

13. $\frac{a^2}{2}+\frac{a}{x}-\frac{x}{a}.$ 14. $x^2-x+\frac{1}{4}.$ 15. $\frac{x^2}{2}-2x+\frac{a}{3}.$

16. $\frac{3a}{x}-\frac{1}{5}+\frac{2x}{3a}.$ 17. $4m^2+\frac{2}{3}n+1.$ 18. $2x^2+8+\frac{8}{x^2}.$

XVI. d. Page 110.

1. $a+1.$ 2. $x+2.$ 3. $ax-y^2.$
4. $2m-1.$ 5. $4a-3b.$ 6. $1+x+x^2.$
7. $1-2x+3x^2.$ 8. $a+2b-c.$ 9. $2a^2-3a+1.$
10. $y^2-y+1.$ 11. $2x^2+x-3.$ 12. $3x^2-2xa+3a^2.$
13. $3x^2-x-1.$ 14. $x^2-2xy+4y^2.$ 15. $3x^2-x+6.$

XVI. e. Page 111.

1. $\frac{x}{2}-1.$ 2. $\frac{x}{3}+2.$ 3. $2x-\frac{y^2}{3}.$

4. $\frac{3x}{4y}-2.$ 5. $x-\frac{3}{x}.$ 6. $\frac{x^2}{y}-2y^2.$

7. $\frac{x}{y}+2-\frac{y}{x}.$ 8. $\frac{x}{3}-1+\frac{3}{x}.$ 9. $\frac{x}{a}-4+\frac{2a}{x}.$

10. $\frac{4a}{x}-4+\frac{x}{a}.$ 11. $\frac{a}{b}-1+\frac{b}{a}.$ 12. $\frac{2x^2}{y^2}+\frac{4x}{y}-3.$

XVII. a. Page 112.

1. $a(a^2-x).$ 2. $x^2(x-1).$ 3. $2a(1-a).$ 4. $a(a-b^2).$
5. $p(7p+1).$ 6. $2x(4-x).$ 7. $5ax(1-a^2x).$
8. $x^2(3+x^3).$ 9. $x(x+y).$ 10. $x^2(x-y).$
11. $5x(1-5xy).$ 12. $5(3+5x^2).$ 13. $16x(1+4xy).$
14. $15a^2(1-15a^2).$ 15. $27(2-3x).$ 16. $5x^3(2-5xy).$
17. $x(3x^2-x+1).$ 18. $2x^3(3+x+2x^2).$ 19. $x(x^2-xy+y^2).$
20. $3a^2(a^2-ab+2b^2).$ 21. $2xy^2(xy-3x+y).$
22. $3x(2x^2-3xy+4y^2).$ 23. $5x^3(x^2-2a^2-3a^3).$
24. $7a(1-a^2+2a^3).$ 25. $19a^3x^2(2x^3+3a).$

XVII. b. Page 113.

1. $(a+b)(a+c)$. **2.** $(a-c)(a+b)$. **3.** $(ac+d)(ac+b)$.
4. $(a+3)(a+c)$. **5.** $(2+c)(x+c)$. **6.** $(x-a)(x+5)$.
7. $(5+b)(a+b)$. **8.** $(a-y)(b-y)$. **9.** $(a-b)(x-z)$.
10. $(p+q)(r-s)$. **11.** $(x-y)(m-n)$. **12.** $(x-a)(m+n)$.
13. $(2x+y)(a+b)$. **14.** $(3a-b)(x-y)$. **15.** $(2x+y)(3x-a)$.
16. $(x-2y)(m-n)$. **17.** $(ax-3by)(x-y)$. **18.** $(x+my)(x-4y)$.
19. $(a+b)(x^2+2)$. **20.** $(x-3)(x-y)$. **21.** $(2x-1)(x^3+2)$.
22. $(3x+5)(x^2+1)$. **23.** $(x+1)(x^3+2)$. **24.** $(y-1)(y^2+1)$.
25. $(a+bc)(xy-z)$. **26.** $(f^2+g^2)(x^2-a)$. **27.** $(2x+3y)(ax-by)$.
28. $(ax+by)(mx-ny)$. **29.** $(a-b-c)(x-y)$. **30.** $(a+b)(ax+by+c)$.

XVII. c. Page 115.

1. $(a+1)(a+2)$. **2.** $(a+1)(a+1)$. **3.** $(a+3)(a+4)$.
4. $(a-4)(a-3)$. **5.** $(x-5)(x-6)$. **6.** $(x-7)(x-8)$.
7. $(x-9)(x-10)$. **8.** $(x+6)(x+7)$. **9.** $(x-10)(x-11)$.
10. $(x-9)(x-12)$. **11.** $(x-5)(x-16)$. **12.** $(x+6)(x+15)$.
13. $(x-7)(x-12)$. **14.** $(x-6)(x-13)$. **15.** $(x-3)(x-15)$.
16. $(x+8)(x+12)$. **17.** $(x-11)(x-15)$. **18.** $(x-13)(x-8)$.
19. $(x+17)(x+6)$. **20.** $(a-19)(a-5)$. **21.** $(a-16)(a-16)$.
22. $(a+15)(a+15)$. **23.** $(a+27)(a+27)$. **24.** $(a-19)(a-19)$.
25. $(a-7b)(a-7b)$. **26.** $(a+2b)(a+3b)$. **27.** $(m-5n)(m-8n)$.
28. $(m-7n)(m-15n)$. **29.** $(x-11y)(x-12y)$. **30.** $(x-13y)(x-13y)$.
31. $(x^2+1)(x^2+7)$. **32.** $(x^2+2y^2)(x^2+7y^2)$. **33.** $(xy-3)(xy-13)$.
34. $(x+24y)(x+25y)$. **35.** $(xy+17)(xy+17)$. **36.** $(a^2b^2+25)(a^2b^2+12)$.
37. $(a-5bx)(a-15bx)$. **38.** $(x+13y)(x+30y)$. **39.** $(a-2b)(a-27b)$.
40. $(x^2+81)(x^2+81)$. **41.** $(4-x)(3-x)$. **42.** $(5+x)(4+x)$.
43. $(12-x)(11-x)$. **44.** $(8+x)(11+x)$. **45.** $(26+xy)(5+xy)$.
46. $(13-xa)(11-xa)$. **47.** $(17-x^2)(12-x^2)$. **48.** $(27+x)(8+x)$.

XVII. d. Page 116.

1. $(x+1)(x-2)$. **2.** $(x+2)(x-1)$. **3.** $(x+2)(x-3)$.
4. $(x+3)(x-2)$. **5.** $(x+1)(x-3)$. **6.** $(x+3)(x-1)$.
7. $(x+8)(x-7)$. **8.** $(x+8)(x-5)$. **9.** $(x+2)(x-6)$.
10. $(a+4)(a-5)$. **11.** $(a+3)(a-7)$. **12.** $(a+5)(a-4)$.
13. $(a+9)(a-13)$. **14.** $(x+12)(x-3)$. **15.** $(x+13)(x-12)$.
16. $(x+11)(x-10)$. **17.** $(x+6)(x-15)$. **18.** $(x+15)(x-16)$.

19. $(a+5)(a-17)$. **20.** $(a+8)(a-19)$. **21.** $(xy+3)(xy-8)$.
22. $(x+12y)(x-5y)$. **23.** $(x+7a)(x-6a)$. **24.** $(x+3y)(x-35y)$.
25. $(a+14y)(a-15y)$. **26.** $(x+23)(x-5)$. **27.** $(x+4y)(x-24y)$.
28. $(x+26)(x-10)$. **29.** $(a+2)(a-13)$. **30.** $(ay+24)(ay-10)$.
31. $(a^2+7b^2)(a^2-8b^2)$. **32.** $(x^2+3)(x^2-17)$. **33.** $(y^2+9x^2)(y^2-3x^2)$.
34. $(ab+2c)(ab-5c)$. **35.** $(a+14bx)(a-2bx)$.
36. $(a+9xy)(a-27xy)$. **37.** $(x^2+25a^2)(x^2-12a^2)$.
38. $(x^2+11a^2)(x^2-12a^2)$. **39.** $(x^2+21a^2)(x^2-22a^2)$.
40. $(x^3+30)(x^3-29)$. **41.** $(1+x)(2-x)$. **42.** $(2+x)(3-x)$.
43. $(11+x)(10-x)$. **44.** $(20+x)(19-x)$. **45.** $(15+ax)(8-ax)$.
46. $(5+xy)(13-xy)$. **47.** $(14+x)(7-x)$. **48.** $(17+x)(12-x)$.

XVII. e. PAGE 119.

1. $(x+1)(2x+1)$. **2.** $(x+1)(3x+2)$. **3.** $(x+2)(2x+1)$.
4. $(x+3)(3x+1)$. **5.** $(x+4)(2x+1)$. **6.** $(x+2)(3x+2)$.
7. $(x+2)(2x+3)$. **8.** $(x+5)(2x+1)$. **9.** $(x+3)(3x+2)$.
10. $(x+2)(5x+1)$. **11.** $(x+2)(2x-1)$. **12.** $(x+1)(3x-2)$.
13. $(x+3)(4x-1)$. **14.** $(x+5)(3x-1)$. **15.** $(x+8)(2x-1)$.
16. $(2x+1)(x-1)$. **17.** $(x+3)(3x-2)$. **18.** $(x+4)(2x-7)$.
19. $(x+6)(3x-5)$. **20.** $(2x+3)(3x-1)$. **21.** $(3x+1)(2x-3)$.
22. $(3x+4)(x+1)$. **23.** $(x+7)(3x+2)$. **24.** $(2x+5)(x-3)$.
25. $(x+7)(3x-2)$. **26.** $(x-7)(3x+2)$. **27.** $(3x-5)(2x-7)$.
28. $(4x-7)(x+2)$. **29.** $(x-2)(3x-7)$. **30.** $(x+13)(3x+2)$.
31. $(x+5)(4x+3)$. **32.** $(2x+y)(x-3y)$. **33.** $(2x-7)(4x-5)$.
34. $(3x-2y)(4x-5y)$. **35.** $(15x-1)(x+15)$. **36.** $(15x-2)(x-5)$.
37. $(12x+5)(x-3)$. **38.** $(12x-7)(2x+3)$. **39.** $(8x-9)(9x-8)$.
40. $(8x+y)(3x-4y)$. **41.** $(2+x)(1-2x)$. **42.** $(3-x)(1+4x)$.
43. $(2+3x)(3-2x)$. **44.** $(4+3x)(1-2x)$. **45.** $(1+7x)(5-3x)$.
46. $(7+3x)(1+x)$. **47.** $(6-x)(3-5x)$. **48.** $(4+5x)(2-x)$.
49. $(5+4x)(4-5x)$. **50.** $(8-9x)(3+8x)$.

XVII. f. PAGE 120.

1. $(x+2)(x-2)$. **2.** $(a+9)(a-9)$. **3.** $(y+10)(y-10)$.
4. $(c+12)(c-12)$. **5.** $(3+a)(3-a)$. **6.** $(7+c)(7-c)$.
7. $(11+x)(11-x)$. **8.** $(20+a)(20-a)$. **9.** $(x+3a)(x-3a)$.
10. $(y+5x)(y-5x)$. **11.** $(6x+5b)(6x-5b)$. **12.** $(3x+1)(3x-1)$.
13. $(6p+7q)(6p-7q)$. **14.** $(2k+1)(2k-1)$. **15.** $(7+10k)(7-10k)$.
16. $(1+5x)(1-5x)$. **17.** $(a+2b)(a-2b)$. **18.** $(3x+y)(3x-y)$.

19. $(pq+6)(pq-6)$. **20.** $(ab+2cd)(ab-2cd)$.
21. $(x^2+3)(x^2-3)$. **22.** $(3a^2+11)(3a^2-11)$.
23. $(5x+8)(5x-8)$. **24.** $(9a^2+7x^2)(9a^2-7x^2)$.
25. $(x^3+5)(x^3-5)$. **26.** $(1+6a^3)(1-6a^3)$. **27.** $(3x^2+a)(3x^2-a)$.
28. $(9x^3+5a)(9x^3-5a)$. **29.** $(x^2a+7)(x^2a-7)$.
30. $(a+8x^3)(a-8x^3)$. **31.** $(ab+3x^3)(ab-3x^3)$. **32.** $(x^3y^3+2)(x^3y^3-2)$.
33. $(1+ab)(1-ab)$. **34.** $(2+x)(2-x)$. **35.** $(3+2a)(3-2a)$.
36. $(3a^2+5b^2)(3a^2-5b^2)$. **37.** $(x^2+4b)(x^2-4b)$.
38. $(x+5y)(x-5y)$. **39.** $(1+10b)(1-10b)$. **40.** $(5+8x)(5-8x)$.
41. $(11a+9x)(11a-9x)$. **42.** $(pq+8a^2)(pq-8a^2)$.
43. $(8x+5z^3)(8x-5z^3)$. **44.** $(7x^2+4y^2)(7x^2-4y^2)$.
45. $(9p^2z^3+5b)(9p^2z^3-5b)$. **46.** $(4x^8+3y^3)(4x^8-3y^3)$.
47. $(6x^{18}+7a^7)(6x^{18}-7a^7)$. **48.** $(1+10a^3b^2c)(1-10a^3b^2c)$.
49. $(5x^5+4a^4)(5x^5-4a^4)$. **50.** $(ab^2c^3+x^8)(ab^2c^3-x^8)$.
51. $1000\times150=150000$. **52.** $241\times1=241$.
53. $1000\times500\times500000$. **54.** $658\times20=13160$.
55. $1006\times500=503000$. **56.** $200\times2=400$.
57. $2000\times1446=2892000$. **58.** $2378\times900=2140200$.
59. $2500\times1122=2805000$. **60.** $3000\times2462=7386000$.
61. $16264\times2=32528$. **62.** $10002\times10000=100020000$.

XVII. g. Page 121.

1. $(a+b+c)(a+b-c)$. **2.** $(a-b+c)(a-b-c)$.
3. $(x+y+2z)(x+y-2z)$. **4.** $(x+2y+a)(x+2y-a)$.
5. $(a+3b+4x)(a+3b-4x)$. **6.** $(x+5a+3y)(x+5a-3y)$.
7. $(x+5c+1)(x+5c-1)$. **8.** $(a-2x+b)(a-2x-b)$.
9. $(2x-3a+3c)(2x-3a-3c)$. **10.** $(a+b-c)(a-b+c)$.
11. $(x+y+z)(x-y-z)$. **12.** $(2a+y-z)(2a-y+z)$.
13. $(3x+2a-3b)(3x-2a+3b)$. **14.** $(1+a-b)(1-a+b)$.
15. $(c+5a-3b)(c-5a+3b)$. **16.** $(a+b+c+d)(a+b-c-d)$.
17. $(a-b+x+y)(a-b-x-y)$. **18.** $(7x+y+1)(7x+y-1)$.
19. $(a+b+m-n)(a+b-m+n)$. **20.** $(a-n+b+m)(a-n-b-m)$.
21. $(b-c+a-x)(b-c-a+x)$. **22.** $(4a+x+b+y)(4a+x-b-y)$.
23. $(a+2b+3x+4y)(a+2b-3x-4y)$.
24. $(1+7a-3b)(1-7a+3b)$. **25.** $(a-b+x-y)(a-b-x+y)$.
26. $(a-3x+4y)(a-3x-4y)$. **27.** $(2a-5x+1)(2a-5x-1)$.
28. $(a+b-c+x-y+z)(a+b-c-x+y-z)$.
29. $(3a+2b+c+x-2y)(3a+2b-c-x+2y)$. **30.** $y(2x+y)$.

31. $y(2x-y)$. **32.** $(x+5y)(x+y)$. **33.** $47x(x+2y)$.
34. $(8x+y)(2x+3y)$. **35.** $5y(6x-5y)$. **36.** $(12x-1)(2x+7)$.
37. $5a(a+2)$. **38.** $(7a+1)(a-1)$. **39.** $3a(a+2b-2c)$.
40. $x(x-14y+2z)$. **41.** $y(2x+y-16)$. **42.** $a(4x+a-6)$.

XVII. h. Page 122.

1. $(x+y+a)(x+y-a)$. **2.** $(a-b+x)(a-b-x)$.
3. $(x-3a+4b)(x-3a-4b)$. **4.** $(2a+b+3c)(2a+b-3c)$.
5. $(x+a+y)(x+a-y)$. **6.** $(a+y+x)(a+y-x)$.
7. $(x+a+b)(x-a-b)$. **8.** $(y+c-x)(y-c+x)$.
9. $(1+x+y)(1-x-y)$. **10.** $(c+x-y)(c-x+y)$.
11. $(x+y+2xy)(x+y-2xy)$. **12.** $(a-2b+3ac)(a-2b-3ac)$.
13. $(x+y+a+b)(x+y-a-b)$. **14.** $(a-b+c+d)(a-b-c-d)$.
15. $(x-2a+b-y)(x-2a-b+y)$. **16.** $(y+b+a+3x)(y+b-a-3x)$.
17. $(x-1+a+2b)(x-1-a-2b)$. **18.** $(3a-1+x+4d)(3a-1-x-4d)$.
19. $(x-y+a-b)(x-y-a+b)$. **20.** $(a-b+c+d)(a-b-c-d)$.
21. $(2x-3a+c+k)(2x-3a-c-k)$.
22. $(a-5b+3bx-1)(a-5b-3bx+1)$.
23. $(a^2+4x^2+5x^3-3)(a^2+4x^2-5x^3+3)$.
24. $(x^2-a^2+x-3)(x^2-a^2-x+3)$.
25. $(a^2+ab+b^2)(a^2-ab+b^2)$. **26.** $(x^2+2xy+4y^2)(x^2-2xy+4y^2)$.
27. $(p^2+3pq+9q^2)(p^2-3pq+9q^2)$. **28.** $(c^2+cd+2d^2)(c^2-cd+2d^2)$.
29. $(x^2+3xy-y^2)(x^2-3xy-y^2)$.
30. $(2m^2+3mn+n^2)(2m^2-3mn+n^2)$, or $(4m^2-n^2)(m^2-n^2)$.

XVII. k. Page 123.

1. $(x-y)(x^2+xy+y^2)$. **2.** $(x+y)(x^2-xy+y^2)$.
3. $(x-1)(x^2+x+1)$. **4.** $(1+a)(1-a+a^2)$.
5. $(2x-y)(4x^2+2xy+y^2)$. **6.** $(x+2y)(x^2-2xy+4y^2)$.
7. $(3x+1)(9x^2-3x+1)$. **8.** $(1-2y)(1+2y+4y^2)$.
9. $(ab-c)(a^2b^2+abc+c^2)$. **10.** $(2x+3y)(4x^2-6xy+9y^2)$.
11. $(1-7x)(1+7x+49x^2)$. **12.** $(4+y)(16-4y+y^2)$.
13. $(5+a)(25-5a+a^2)$. **14.** $(6-a)(36+6a+a^2)$.
15. $(ab+8)(a^2b^2-8ab+64)$. **16.** $(10y-1)(100y^2+10y+1)$.
17. $(x+4y)(x^2-4xy+16y^2)$. **18.** $(3-10x)(9+30x+100x^2)$.
19. $(ab+6c)(a^2b^2-6abc+36c^2)$. **20.** $(7-2x)(49+14x+4x^2)$.
21. $(a+3b)(a^2-3ab+9b^2)$. **22.** $(3x-4y)(9x^2+12xy+16y^2)$.
23. $(5x-1)(25x^2+5x+1)$. **24.** $(6p-7)(36p^2+42p+49)$.
25. $(xy+z)(x^2y^2-xyz+z^2)$. **26.** $(abc-1)(a^2b^2c^2+abc+1)$.

27. $(7x+10y)(49x^2-70xy+100y^2)$.

28. $(9a-4b)(81a^2+36ab+16b^2)$.

29. $(2ab+5x)(4a^2b^2-10abx+25x^2)$.

30. $(xy-6z)(x^2y^2+6xyz+36z^2)$.
31. $(x^2-3y)(x^4+3x^2y+9y^2)$.

32. $(4x^2+5y)(16x^4-20x^2y+25y^2)$.
33. $(2x-z^2)(4x^2+2xz^2+z^4)$.

34. $(6x^2-b)(36x^4+6x^2b+b^2)$.
35. $(a+7b)(a^2-7ab+49b^2)$.

36. $(a^2+9b)(a^4-9a^2b+81b^2)$.
37. $(2x-9y^2)(4x^2+18xy^2+81y^4)$.

38. $(pq-3x)(p^2q^2+3pqx+9x^2)$.
39. $(z-4y^2)(z^2+4zy^2+16y^4)$.

40. $(xy-8)(x^2y^2+8xy+64)$.

XVII. 1. PAGE 124_A.

1. $(x-1)(x-2)$.
2. $(a+2)(a+5)$.
3. $(b+4)(b-3)$.
4. $(y-7)(y+3)$.
5. $(c+1)(c+11)$.
6. $(x-5)(x+1)$.
7. $(n+2)(n+10)$.
8. $(y+10)(y-1)$.
9. $(p-6q)(p+4q)$.
10. $(y+11)(y-10)$.
11. $(z-15)(z+6)$.
12. $(k-6)(k-8)$.
13. $(a+9)(a+9)$.
14. $(b-27)(b+3)$.
15. $(c+27)(c+3)$.
16. $(x-7)(x-7)$.
17. $(y+7z)(y+3z)$.
18. $(z+9)(z-7)$.
19. $(n+8)(n+3)$.
20. $(p-8)(p+3)$.
21. $(l+12)(l-3)$.
22. $(ab-2)(ab-2)$.
23. $(ab+8)(ab+2)$.
24. $(b-9)(b+5)$.
25. $(m+11)(m-8)$.
26. $(n-15)(n+3)$.
27. $(p+13)(p-3)$.
28. $(xy-9)(xy+8)$.
29. $(z-5)(z+4)$.
30. $(x+8y)(x-7y)$.
31. $(a-13b)(a+2b)$.
32. $(ab-8)(ab+7)$.
33. $(y^2+13)(y^2-12)$.
34. $(z^2-13)(z^2+6)$.
35. $(y^2+5)(y^2-7)$.
36. $(x+13y)(x-7y)$.
37. $m^2n^2(m-3n)$.
38. $5x^3(2+5xy)$.
39. $(y-5)(y+3)$.
40. $(a+b)(x+y)$.
41. $(x+y)(x-z)$.
42. $(3c-2)(c+1)$.
43. $(2b+1)(b+5)$.
44. $(x-3y)(x-3y)$.
45. $(3x-1)(x-3)$.
46. $(cd+1)(cd-2)$.
47. $(2x+3)(3x-1)$.
48. $(a-b)(4-c)$.
49. $(a^3+2)(a+1)$.
50. $2c^2d(c-3d+d^2)$.
51. $xy(x+9)(x-7)$.
52. $(2y-3)(3y+1)$.
53. $(2x-3)(2x-3)$.
54. $(3+4p)(1-3p)$.
55. $(4+pq)(4+pq)$.
56. $z(4z-3)(z+2)$.
57. $a(a+7)(a-6)$.
58. $(m^3+2)(2m-1)$.
59. $a^2(a-b)(a-3)$.
60. $(7+x)(2-x)$.
61. $(17-z)(1-z)$.
62. $(2m^2+3)(m^2-7)$.
63. $(5x-3y)(x+2y)$.
64. $(3m^3-5)(2m^3+9)$.
65. $(3m-4)(3m-4)$.
66. $(5+9a)(5-9a)$.

67. $(a^2b^2+3)(a^2b^2-3)$.
68. $(3+l)(9-3l+l^2)$.
69. $(1-4m)(1+4m+16m^2)$.
70. $(k^2+5l)(k^2-5l)$.
71. $(pq-1)(p^2q^2+pq+1)$.
72. $(2z+1)(4z^2-2z+1)$.
73. $(1+8x)(1-8x)$.
74. $2(5p+1)(25p^2-5p+1)$.
75. $4(5ab^2+1)(5ab^2-1)$.
76. $(9+cd)(81-9cd+c^2d^2)$.
77. $(a+x+1)(a+x-1)$.
78. $(4+b-c)(4-b+c)$.
79. $x(3x+2y)(3x-2y)$.
80. $(p-5q)(p+4q)$.
81. $l(l-7)(l+6)$.
82. $(abc+9d)(abc-9d)$.
83. $(4x^2-3y)(16x^4+12x^2y+9y^2)$.
84. $(x-17)(x+19)$.
85. $(x^2+17)(x^2-17)$.
86. $(l+17)(l-16)$.
87. $(10z-3)(100z^2+30z+9)$.
88. $(a+23)(a-13)$.
89. $(a+b+c)(a-b-c)$.
90. $(1+x-3y)(1-x+3y)$.
91. $(x^2+y^2+3xy)(x^2+y^2-3xy)$.
92. $(a^2+a+2)(a^2-a+2)$.
93. $(b-29)(b+27)$.
94. $(x+2)(x^2-2x+4)(x-2)(x^2+2x+4)$.
95. $(3y+2x)(9y^2-6xy+4x^2)(3y-2x)(9y^2+6xy+4x^2)$.
96. $(x^4+1)(x^2+1)(x+1)(x-1)$.
97. $ab(3a+b)(9a^2-3ab+b^2)(3a-b)(9a^2+3ab+b^2)$.
98. $a^2(ax+2y)(a^2x^2-2axy+4y^2)(ax-2y)(a^2x^2+2axy+4y^2)$.
99. $(a^2+b^2)(a^4-a^2b^2+b^4)(a+b)(a^2-ab+b^2)(a-b)(a^2+ab+b^2)$.
100. $(x^2+2y^2z^2)(x^2+2y^2z^2)$.
101. $(ab+8)(a^2b^2-8ab+64)$.
102. $(2x+7)(x+5)$.
103. $20y(5x+y)(5x-y)$.
104. $\{(a+b)^2+1\}(a+b+1)(a+b-1)$.
105. $(c+d-1)\{(c+d)^2+c+d+1\}$.
106. $(1-x+y)\{1+x-y+(x-y)^2\}$.
107. $(x-19)(x+13)$.
108. $(a+9)(a-31)$.
109. $2\{5(a-b)+1\}\{25(a-b)^2-5(a-b)+1\}$.
110. $2c(c^2+3d^2)$.
111. $9y(4x^2+2xy+y^2)$.
112. $(x-2y)(x+2y+1)$.
113. $(a-b)(a+b+1)$.
114. $(a+b)(a+b+1)$.
115. $(a+b)(a^2-ab+b^2+1)$.
116. $(a+3b)(a-3b+1)$.
117. $(x-y)\{2(x-y)+1\}\{2(x-y)-1\}$.
118. $xy(x+y)(x-y)(x-y)$.

Miscellaneous Examples III. Page 126.

1. x^3-2x. **2.** $42a-40b+30c$. **3.** a^6-c^6.

4. (1) 12; (2) $x=5$, $y=6$. **5.** x^3+4x-1.

6. 72. **7.** $\frac{109}{210}$. **8.** $x^2+\frac{3}{4}x+\frac{5}{4}$.

9. $2x^2-x$. **10.** (1) $(ax-5)(ax+3)$; (2) $(2m^2+9pq)(2m^2-9pq)$.

11. (1) $x=-2$, $y=4$; (2) $x=5$, $y=-2$. **12.** $\frac{am}{pb}$ miles.

13. $84x^4+25x^3+101x-30$. **14.** (1) 7; (2) $-1\frac{1}{2}$.

15. $x^4+14x^3+27x^2-154x+121$.

16. (1) $(x+2a)(x-b)$; (2) $(x^2+14y)(x^2-4y)$.

17. H.C.F. $=7$; L.C.M. $=3528a^3b^2c^3$. **18.** \$14. **19.** $3p$.

21. (1) $m-n=a+c$; (2) $3a^2b^2+c^3=p(m+n)$.

22. 8. **23.** $6x^2-xy-y^2$.

24. Train 31 miles, coach 7 miles.

25. $(x-5)(2x-3)(x+1)$. **26.** 33. **27.** $2x^2+9xy-7y^2$.

28. $x=2$, $y=3$, $z=0$.

29. (1) $xy(x+2y)(x-2y)$; (2) $(m+n)(m-n)(2m^2+3n^2)$.

30. $\frac{bc}{am}$ days, $2\frac{5}{8}$.

XVIII. a. Page 129.

1.	$a+b$.	**2.**	$x+y$.	**3.**	$x(x-y)$.	**4.**	$2x-3y$.
5.	$x+y$.	**6.**	$ab(a-b)$.	**7.**	$a(a-x)$.	**8.**	$a+2x$.
9.	$b(a+b)$.	**10.**	$x-3y$.	**11.**	$a-x$.	**12.**	$2x+y$.
13.	$2(5x-1)$.	**14.**	$3x+2y$.	**15.**	$x+1$.	**16.**	$y(x-1)$.
17.	$(x-y)^2$.	**18.**	x^2+a^2.	**19.**	$x+2y$.	**20.**	$x-3a$.
21.	$x+2$.	**22.**	$x-5$.	**23.**	$x-3$.	**24.**	$x-3$.
25.	$3x+1$.	**26.**	$x-1$.	**27.**	$cx+d$.	**28.**	x^2+y.
29.	$x(a-3b)$.	**30.**	$2x+1$.	**31.**	$x^2(3x+2)$.		

XVIII. b. Page 133.

1. x^2-3x+2.	2. $x^2-13x+5$.	3. x^2-8.
4. x^2-5.	5. x^2+2x+1.	6. $x+3$.
7. $a^2-2ax+x^2$.	8. $x+1$.	9. x^2-3x+7.
10. $2x^2-7$.	11. $3x^2+1$.	12. $2x^2-3$.
13. $3x^2+2a^2$.	14. x^2-ax+a^2.	15. $x^2+2ax-a^2$.
16. $3a^2-ax-2x^2$.	17. $xy(2x^2+xy-3y^2)$.	18. $2x^2a^2(2x-3a)$.
19. $2x^2(2x+7)$.	20. $6(3x-5a)$.	21. $3x^2-2xy+y^2$.
22. x^4+x^3-1.	23. $1+x^3-x^4$.	24. $1+a$.
25. $x(3+4x)$.	26. x^2-2x+1.	27. $2x^2-7$.

XIX. a. Page 137.

1. $\frac{3}{2b}$.	2. $\frac{b}{c}$.	3. $\frac{1}{ax-1}$.	4. $\frac{3b^2c}{20(a-b)}$.
5. $\frac{2x-3y}{2x}$.	6. $4(x-y)$.	7. $\frac{1}{2a+3x}$.	8. $\frac{x}{x^2-2y^2}$.
9. $\frac{x-3y}{x^2+3xy+9y^2}$.	10. $\frac{x}{x+1}$.	11. $\frac{3x}{x+2}$.	12. $\frac{5a}{3b}$.
13. $\frac{xy}{x-2}$.	14. $\frac{3(a+b)}{a-b}$.	15. $\frac{x^2-17}{x^2-5}$.	16. $\frac{x+2y}{x^2+xy+y^2}$.
17. $\frac{2x+3}{3x+5}$.	18. $\frac{a(x-4)}{x+5}$.	19. $\frac{x+7}{x+13}$.	20. $\frac{3+a}{2}$.

XIX. b. Page 139.

The expression in [ ] is in each case the H.C.F. of the numerator and the denominator.

1. $\frac{a-2b}{a+2b}$ $[a^2+ab+b^2]$.	2. $\frac{x-3}{x+2}$ $[(x-1)^2]$.
3. $\frac{a+5}{a+4}$ $[(a-1)(a-2)]$.	4. $\frac{x^2+4xy-9y^2}{2x^2+3xy+7y^2}$ $[2x-3y]$.
5. $\frac{2a+5b}{3a+5b}$ $[(2a+3b)(a-b)]$	6. $\frac{1-x+2x^2}{1-x+3x^2}$ $[1+x+x^2]$.
7. $\frac{x-1}{3x^2+3x+10}$ $[x-1]$.	8. $\frac{3a^2+b^2}{4a-b}$ $[a-b]$.
9. $\frac{4x^2-ax+a^2}{x^3+a^3}$ $[x+a]$.	10. $\frac{2(2x^2-3x-1)}{3x^3+x^2+x-2}$ $[x-1]$.
11. $(2x-3a)^2$ $[(2x+3a)^2]$.	12. $\frac{3x^2-x-2}{3x^2+x-2}$ $[2x+1]$.

13. $\frac{5x+2}{7x-4}$ $[x^2-3]$. **14.** $\frac{2x^2+3x+5}{2x^2+3x-5}$ $[2x^2-3x+5]$.

15. $\frac{3(x-3a)(x-4a)}{2(x+3a)(x+4a)}$ $[x-2a]$. **16.** $\frac{a(x+8a)}{x(x+7a)}$ $[x^2-13ax+5a^2]$.

XIX. c. Page 142.

1. $\frac{7}{12}$. **2.** $\frac{ab}{2a-1}$. **3.** 2. **4.** $\frac{a-11}{a-2}$.

5. $\frac{4x+3a}{x+2}$. **6.** $\frac{5a-b}{x(3a-2)}$. **7.** $\frac{x+2}{x-1}$. **8.** $\frac{x+1}{x+5}$.

9. $\frac{x}{x-2}$. **10.** $\frac{2x-1}{2x-3}$. **11.** $\frac{x+1}{x+5}$. **12.** $\frac{x-1}{4x+7}$.

13. b^2+3b+9. **14.** $\frac{1}{x+7}$. **15.** $8pq-z^2$. **16.** x.

17. $\frac{x+1}{x-1}$. **18.** $\frac{x-5}{x-1}$. **19.** $\frac{x}{y}$. **20.** x.

21. $\frac{2x-1}{2x-5}$. **22.** 1. **23.** $\frac{1}{b}$. **24.** $\frac{a+b-c}{b-c-a}$.

25. $\frac{1}{x-8}$. **26.** $\frac{a-x}{a+x}$. **27.** $\frac{m}{n}$. **28.** $x(2+x)$.

29. $\frac{x+4}{x(x-4)}$. **30.** 1. **31.** $a+x$. **32.** $\frac{a^2}{16a^2+4ab+b^2}$.

XX. a. Page 144.

1. $x(x+1)$. **2.** $x^2(x-3)$. **3.** $12x^2(x+2)$. **4.** $21x^3(x+1)$.

5. $x(x+1)(x-1)$. **6.** $ab(a+b)$. **7.** $xy(2x+1)(2x-1)$.

8. $6x(3x-1)$. **9.** $x(x+1)(x+2)$. **10.** $(x+1)(x-1)(x-2)$.

11. $(x+2)^2(x+3)$. **12.** $(x-1)(x-2)(x-4)$.

13. $(x-3)(x-1)(x+2)$. **14.** $(x+5)(x-4)(x-6)$.

15. $(x+7)(x-6)(x-5)$. **16.** $(x+1)(x+2)(2x+1)$.

17. $(x+2)(x+3)(3x+2)$. **18.** $(x+2)(x+3)(5x+1)$.

19. $(x+2)(x+8)(2x-1)$. **20.** $(x+2)(x-2)(3x-7)$.

21. $12x(x+2)(2x+1)(4x-7)$. **22.** $6x^2(x+7)(3x+5)(3x-2)$.

23. $20x^2y(3x+1)(5x+1)(4x-1)$. **24.** $(x+y)(2x-7y)(4x-5y)$.

25. $(x-y)(3x-2y)(4x-5y)$. **26** $3a^2x(3x-a)(2x+3a)(x+5a)$.

27. $2axy^3(x+3)(4x-1)(3x-2)$. **28.** $x^2(3-5x)^2(2+x)^2$.

29. $42a^4b^2(a-b)^3(a+b)(a^2+ab+b^2)$.

30. $m^3n(m^6-n^3)(m-n)^2$. **31.** $8c^2(2c-3d)^2(8c^3-27d^3)$.

XX. b. Page 146.

1. H.C.F. $x-2$. L.C.M. $(x+1)^2(x+2)(x-2)(x-3)$.
2. $(ax+b)(ax-b)(bx+a)$. 3. $xy(x-a)(y-b)(y-2b)$.
4. H.C.F. $x(x+3)$. L.C.M. $x(x-1)(x+3)(2x-1)$.
5. $(1+x)^3(1-x)^2$. 6. $(x-2)(x-4)(x-6)$.
7. H.C.F. $2x+1$. L.C.M. $(2x+1)(x+1)(x-1)(3x+2)(3x-2)$.
8. $ab^2c^2(c+a)^2(c-a)^2$.
9. L.C.M. $y^2(x-y)^2(x^2+xy+y^2)$. H.C.F. $x-y$.
10. H.C.F. $2x-3$. L.C.M. $(2x-3)(3x-2)(x+4)(3x+4)$.
11. $(x+a)^2(x^2+ax+a^2)(x^2-ax+a^2)$.
12. H.C.F. $3x-y$. L.C.M. $(3x-y)(x+y)^2(x-y)^2$.
13. $x-1$. 14. $(a+b)(a-b)(a-2b)(a^2+ab+b^2)$.
15. H.C.F. a^2+xy. L.C.M. $(a^2+xy)(2x+3y)(2x-3y)$.
16. H.C.F. $(x-3)(x-4)$. L.C.M. $(x-2)(x-3)(x-4)(x-5)$.
17. $x-8a$. 18. $105x^2y^2(x+y)^2(x-y)^2$.

XXI. a. Page 150.

1. $\frac{4(x+1)}{5}$. 2. $\frac{13(x-2)}{12}$. 3. $\frac{25x-61}{56}$. 4. $\frac{17x}{36}$.
5. $\frac{19x-201}{225}$. 6. $\frac{12x^2+28x-27}{8x^2}$. 7. 0. 8. $\frac{3(a+3b)}{8a}$.
9. $\frac{6b^2c+6bc^2+3ac^2+3a^2c-4a^2b+4ab^2}{12abc}$. 10. $\frac{a^2+3x^2}{2ax}$.
11. $\frac{5x+31}{102x}$. 12. $\frac{a^4b^2-b^4c^2+a^2c^4}{a^2b^2c^2}$. 13. $\frac{11x^3-18x^2-27x-16}{30x^3}$.
14. $\frac{x^3+y^3}{x^2y^3}$. 15. $\frac{3y+2z}{yz}$. 16. $\frac{a^3+b^3+c^3-3abc}{abc}$.

XXI. b. Page 151.

1. $\frac{2x+5}{(x+2)(x+3)}$. 2. $\frac{x+5}{(x+3)(x+4)}$. 3. $\frac{1}{(x-4)(x-5)}$.
4. $\frac{2(x+6)}{(x-6)(x+2)}$. 5. $\frac{(a-b)x}{(x+a)(x+b)}$. 6. $\frac{(a+b)x-2ab}{(x-a)(x-b)}$.
7. $\frac{2}{(x+2)(x+4)}$. 8. $\frac{4ax}{a^2-x^2}$. 9. $\frac{8x}{x^2-4}$.
10. $\frac{6}{(x-2)(x-5)}$. 11. $\frac{ax}{x^2-a^2}$. 12. $\frac{5x+9}{x^2-9}$.
13. $\frac{x+2y}{4x^2-9y^2}$. 14. $\frac{3ax}{x^2-4a^2}$. 15. $\frac{4ab}{4a^2-b^2}$.

16. $\dfrac{2xy}{x^2-y^2}$. 17. $\dfrac{2x^3}{1-x^4}$. 18. $\dfrac{x^2+y^2}{xy(x^2-y^2)}$.

19. $\dfrac{5x^2}{25x^2-y^2}$. 20. $\dfrac{x^4+y^4}{xy(x^4-y^4)}$. 21. $\dfrac{4a^2}{x(x+2a)}$.

22. $\dfrac{2x^3}{x^2-y^2}$. 23. $-\dfrac{2ax}{a^3-8x^3}$. 24. $2b$.

25. $\dfrac{4(x-1)}{(x-2)^2(x+2)}$. 26. $\dfrac{x^2+a^2}{ax(x-a)(x+a)^2}$.

XXI. c. Page 153.

1. $\dfrac{2}{x+y}$. 2. $\dfrac{x}{4x^2-y^2}$. 3. $\dfrac{1-6x^2}{1-4x^2}$. 4. $\dfrac{4a^2+b^2}{4a^2-9b^2}$.

5. $\dfrac{1+a}{9-a^2}$. 6. $\dfrac{4x-5}{6(x^2-1)}$. 7. 0. 8. $\dfrac{12a^2-4a+7}{3(4a^2-9)}$.

9. $\dfrac{2(13x+7)}{3(x^2-4)}$. 10. $\dfrac{x^2+y^2}{x^4+x^2y^2+y^4}$. 11. $\dfrac{2}{(x-4)(x-6)}$.

12. $\dfrac{2}{(x-2)(x-3)(x-4)}$. 13. $\dfrac{2}{(x-1)(2x+1)(2x+3)}$.

14. $\dfrac{1}{(x-1)(2x+1)(3x-2)}$. 15. $\dfrac{17a}{(1-2a)(4+a)(3+5a)}$.

16. $\dfrac{23x}{(1+2x)(2+x)(5-9x)}$. 17. $\dfrac{x+2}{(x+1)(x+3)}$.

18. $\dfrac{1}{x+1}$. 19. $\dfrac{1}{a+b}$. 20. $\dfrac{3x+2}{(x-2)(x-1)(x+1)}$.

21. $\dfrac{1}{x+2}$. 22. $\dfrac{8x^2+4x-3}{(x-1)(x+1)(2x+1)}$. 23. $\dfrac{1}{2x+1}$.

24. $\dfrac{x^2+11}{(x-1)(x+2)(x+3)}$. 25. $\dfrac{2x+13}{(x+3)(x+4)(x-4)}$.

26. $\dfrac{32a^2}{(1-2a)^2(1+2a)}$. 27. $\dfrac{96x^2}{(3-2x)^2(3+2x)}$.

28. $\dfrac{4x^3}{81-x^4}$. 29. $\dfrac{72a}{16a^4-81}$. 30. $\dfrac{1}{1-x^4}$.

31. $\dfrac{a(a^2+2ax+3x^2)}{4(a^4-x^4)}$. 32. $\dfrac{16x}{16-x^4}$. 33. $\dfrac{x(37+172x^2)}{6(1-16x^4)}$.

34. $\dfrac{2a(a^2+32x^2)}{3(a^4-256x^4)}$. 35. $\dfrac{7a^2+45}{6(a^4-81)}$. 36. $\dfrac{x^5}{1-x^8}$.

37. $\dfrac{36a^4}{a^8-6561}$. 38. $\dfrac{2}{x^2(x^2-4)}$. 39. $\dfrac{1}{(3x-y)(x-3y)}$.

40. $\dfrac{2}{(x-1)(x+1)^2}$. 41. 1. 42. 0. 43. $\dfrac{4x}{x^2-1}$.

XXI. d. Page 156.

1. $\frac{x-11}{20(x^2-1)}$. 2. $\frac{1}{1-a^2}$. 3. $\frac{x+3a}{x+a}$. 4. $\frac{2x-a}{x+a}$.

5. 0. 6. $\frac{7x}{1-x^2}$. 7. $\frac{1}{x-3}$. 8. $\frac{12(2x+1)}{4x^2-9}$.

9. $\frac{61-21b}{12(1-b^2)}$. 10. $\frac{2}{3(1-a^2)}$. 11. $\frac{-y^5}{-x^6+y^6}$. 12. $\frac{x}{y}$.

13. $\frac{2x}{x+y}$. 14. $\frac{2x^3}{x^2-4}$. 15. $\frac{a}{4a^2-25b^2}$. 16. $\frac{b(a+b)}{x^2-b^2}$.

17. $\frac{2bx}{4x^2-1}$. 18. $\frac{x+c}{(x-a)(x-b)}$. 19. $\frac{x-c}{(x-a)(x-b)}$.

20. $\frac{2a}{(x-a)(x-b)}$. 21. 0. 22. $\frac{4a^3}{x^4-a^4}$. 23. $\frac{48a^3}{(x^2-a^2)(x^2-9a^2)}$.

24. $\frac{x^4}{a^8-x^8}$. 25. 0. 26. $\frac{a^6}{a^8-b^8}$. 27. $\frac{a-x}{a+x}$.

28. $\frac{a^3}{(a-b)(a^3+b^3)}$. 29. $\frac{2+x+3x^2}{2(1-x^4)}$. 30. $\frac{2(x^2+1)}{x(x^2-1)}$.

31. 0. 32. $\frac{4ab}{a^2-b^2}$.

XXI. e. Page 159.

1. 0. 2. $\frac{bc+ca+ab-a^2-b^2-c^2}{(a-b)(b-c)(c-a)}$. 3. $\frac{x^2+y^2+z^2-yz-zx-xy}{(x-y)(y-z)(z-x)}$.

4. 0. 5. $\frac{2(bc+ca+ab-a^2-b^2-c^2)}{(a-b)(b-c)(c-a)}$ 6. 0.

7. 0. 8. 0. 9. $\frac{2(qr+rp+pq-p^2-q^2-r^2)}{(p-q)(q-r)(r-p)}$.

10. 0. 11. 0. 12. $\frac{p(y-z)+q(z-x)+r(x-y)}{(y-z)(z-x)(x-y)}$.

XXII. a. Page 163.

1. $\frac{m^2-nl}{na-mb}$. 2. $\frac{x+y}{y-x}$. 3. $\frac{ad+b}{dx-y}$. 4. $\frac{x+c}{b-x}$.

5. $\frac{3}{4b}$. 6. $\frac{a}{c}$. 7. $\frac{x^2-y^2}{x^2+y^2}$. 8. $\frac{c}{ac+b}$.

9. $\frac{ad}{bd+c}$. 10. $\frac{nx}{nx-m}$. 11. $\frac{pn(ad+bc)}{bd(pm+kn)}$. 12. $x-1$.

13. $\frac{x(x+3)}{x+4}$. 14. $-\frac{x+1}{x^2(x+3)}$. 15. $-\frac{x^2(2x+3)}{x+2}$. 16. $\frac{1}{x}$.

17. $\dfrac{a^2-b^2}{2}$. 18. 2. 19. $\dfrac{y^4}{x^2+y^2}$. 20. $\dfrac{1}{2x^2-1}$.

21. $\dfrac{2(a+b)}{a-b}$. 22. $a+x$. 23. $\dfrac{4}{x^2}$. 24. $\dfrac{1+x}{1+x^2}$.

25. $\dfrac{a^2-a+1}{2a-1}$. 26. $\dfrac{6+x+2y}{8x(y+6)}$. 27. $\dfrac{a(yz+n)}{xyz+nx+mz}$. 28. $1-x$.

29. $\dfrac{x^2-3x+1}{x^2-4x+1}$. 30. $\dfrac{2}{x^3}$. 31. $\dfrac{a-c}{1+ac}$. 32. $\dfrac{b}{a}$.

33. $\dfrac{1}{a+x}$. 34. 4. 35. $8x^2-1$. 36. $2x^2$.

XXII. b. Page 167.

1. $\dfrac{x}{3}+\dfrac{y}{9}-\dfrac{y^2}{9x}$. 2. $\dfrac{a^2}{4}-\dfrac{ax}{3}+\dfrac{x^2}{2}$. 3. $\dfrac{a^2}{2b}-\dfrac{3a}{2}+\dfrac{3b}{2}+\dfrac{b^2}{2a}$.

4. $\dfrac{1}{bc}+\dfrac{1}{ca}+\dfrac{1}{ab}$. 5. $\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}$. 6. $\dfrac{a^2}{6}-\dfrac{b^2}{2}+\dfrac{1}{3}$.

7. $x-x^2+x^3-x^4$; Rem. x^5. 8. $1+\dfrac{b}{a}+\dfrac{b^2}{a^2}+\dfrac{b^3}{a^3}$; Rem. $\dfrac{b^4}{a^3}$.

9. $1+2x+2x^2+2x^3$; Rem. $2x^4$. 10. $1+x-x^3-x^4$; Rem. x^6.

11. $x-3+\dfrac{9}{x}-\dfrac{27}{x^2}$; Rem. $\dfrac{81}{x^2}$. 12. $1+2x+3x^2+4x^3$; Rem. $5x^4-4x^5$.

17. $\dfrac{x-3}{x-4}$. 18. $3(a-2x)^2$. 19. $\dfrac{b^2-3b-2}{b-6}$.

20. $\dfrac{a^2-4b^2}{a+3b}$. 21. $\dfrac{(2x-3)(2x+7)}{6}$.

XXII. c. Page 168.

1. $\dfrac{4(c-x)}{3(a+x)}$. 2. $\dfrac{x(x+a)}{2}$. 3. $\dfrac{1}{a^2+ab-2b^2}$.

4. $\dfrac{8xy(x^2+y^2)}{(x^2-y^2)^2}$. 5. $\dfrac{4x(2-x)}{(x-1)(x^3+1)}$. 6. $\dfrac{x^4}{1-x^8}$.

7. $\dfrac{1}{x(x+1)^2(1+x+x^2)}$. 8. $\dfrac{1-x+x^2}{1+x+x^2}$.

9. $\dfrac{2x+3}{3(x+6)}$. 10. $\dfrac{bx+a}{ax+b}$. 11. $\dfrac{ax^3(x^2+a^2)}{x^3+a^3}$.

12. $\dfrac{a^6}{(a-x)(a+x)^2}$. 13. $\dfrac{(x+1)^2}{3x^3+6x^2-x-8}$. 14. $\dfrac{a+y}{a^2-y}$.

15. $\dfrac{2(a^2+a+1)}{a(a+1)(a+2)}$. 16. $\dfrac{1}{2(3-2x)}$. 17. $\dfrac{4}{1-x^4}$.

18. 1. **19.** $\dfrac{(2x-1)(x+1)}{(x+2)(x-1)}$. **20.** $\dfrac{x-2}{4x^2-5x-5}$.

21. $\dfrac{x}{(x-2a)^2}$. **22.** $\dfrac{a(a^2+x^2)}{(x-a)(a+x)^2}$. **23.** 1. **24.** 1.

25. $9x-\dfrac{1}{x}$. **26.** $\dfrac{a}{2x^2}$. **27.** $\dfrac{1}{2x(2x-1)}$. **28.** $\dfrac{b^4}{b^2+a^2}$.

29. $\dfrac{ab}{a+b}$. **30.** x. **31.** x. **32.** bx. **33.** $\dfrac{a^2-b^2}{2}$.

34. 1. **35.** $\left(x-\dfrac{1}{x}\right)^2$. **36.** 1. **37.** 1.

38. $\dfrac{x(x+1)}{x^2+4x+1}$. **39.** $\dfrac{12}{(a^4-4)(a^4-1)}$. **40.** $\dfrac{3n^2}{(3m+2n)(9m^2-n^2)}$

41. $\dfrac{1+x+x^2}{(1+x)(1+x^2)(1-x)^2}$. **42.** $\dfrac{2x}{(x-2)(x+1)^2}$. **43.** $\dfrac{1}{x+y}$.

44. 1. **45.** 1. **46.** 1. **47.** 0.

48. $\dfrac{a^2+b^2+c^2-bc-ca-ab}{(b-c)(c-a)(a-b)}$. **49.** 1. **50.** 1. **51.** 0.

52. $2y+a+b$. **53.** $\dfrac{(2a^2+x^2)(a-x)}{a^2x}$. **54.** $\dfrac{28(x+4)}{9(x+3)}$.

55. $\dfrac{7(x-4)}{4(x-1)}$. **56.** $x+3$. **57.** $1+a-a^3$. **58.** $-\dfrac{c}{c}$.

Miscellaneous Examples IV. Page 172.

1. $-\dfrac{1}{22}$. **2.** 6. **3.** $abc(b-c)$; -6. **4.** 7. **5.** $\dfrac{5}{3}$.

6. (1) 232; (2) -29. **7.** (1) -19; (2) 0. **8.** 1. **9.** $-\dfrac{3}{10}$.

10. (1) -12; (2) 1. **11.** 1. **12.** $8\frac{1}{2}$. **13.** $98x-2y$; $19\frac{1}{3}$.

14. (1) 1; (2) 21. **15.** $(x+9)(x+12)$. **16.** $(a-7)(a+13)$.

17. $(x-8y)(x-12y)$. **18.** $(ab-17)(ab+3)$. **19.** $c(c+13)(c-12)$.

20. $n(m-3n)(m-3n)$. **21.** $(p^2+7q^2)(p^2-8q^2)$.

22. $(d^2+5c^2)(d+3c)(d-3c)$. **23.** $xy(x+6y)(x-7y)$.

24. $(m+13)(m+15)$. **25.** $(14-a)(15+a)$. **26.** $(19-pq)(3+pq)$.

27. $(x^2+16)(x^2+11)$. **28.** $(a^2+14)(a^2-7)$. **29.** $(c+27)(c+27)$.

30. $(9-xy)(8+xy)$. **31.** $(a^2+2x^2)(a^2+7x^2)$.

32. $(p-12q)(p+9q)$. **33.** $2(a^3+12)(a^3-11)$.

34. $x^2(x-9)(x+7)$. **35.** $(bc+12)(bc-7)$. **36.** $(z+17)(z+17)$.

37. $(a-3c)(a-19c)$. **38.** $yz(y-7)(y+13)$.

39. $(2+3x^3)(1-x)(1+x+x^2)$. **40.** $(2ab-5)(ab+3)$.

41. $(3p-4)(3p-4)$. **42.** $(5+mn)(7+mn)$. **43.** $(17+c)(7-c)$.
44. $x^3(2-x)(3-x)$. **45.** $(2m+3)(3m-1)$. **46.** $(2a-5b)(2a+b)$.
47. $(6p-q)(p-2q)$. **48.** $(5x+4z)(4x-5z)$. **49.** $(2x^2+3)(4x^2-5)$.
50. $6(2y-1)(y-2)$. **51.** $(3ab+4)(4ab-3)$.
52. $(2a^2b-5)(a^2b-2)$. **53.** $(7x+8y)(3x-2y)$.
54. $(9m-5n)(2m+3n)$. **55.** $(c+a-b)(c-a+b)$.
56. $(a+b-c)(a-b+c)$. **57.** $(5x+3y)(25x^2-15xy+9y^2)$.
58. $(ab+7)(a^2b^2-7ab+49)$. **59.** $(8b-a^2)(64b^2+8ba^2+a^4)$.
60. $(a+2x-2y)(a-2x+2y)$. **61.** $(m+n+1)(m+n-1)$.
62. $2c^2(3c+d)(c-d)$. **63.** $(a^2b^2-1+x-y)(a^2b^2-1-x+y)$.
64. $(1+2m)(1-2m)(1-2m+4m^2)(1+2m+4m^2)$.
65. $p^3(1+10q)(1-10q+100q^2)$. **66.** $(81+a^2)(9+a)(9-a)$.
67. $(x^2-1+y-z)(x^2-1-y+z)$. **68.** $(a+4b-4c)(a-4b+4c)$.
69. $(c-d)(1+2c-2d)(1-2c+2d)$. **70.** $(p-4q)(p+4q+1)$.
71. $2[1+4a+4b][1-4(a+b)+16(a+b)^2]$.
72. $(x+3y)(1+x^2-3xy+9y^2)$. **73.** $(x+y)(x^2+y^2)$.
74. $(cx-d)(ax+b)$. **75.** $(7+a)(2-a)$.
76. $(14x^2+y^2)(7x^2-y^2)$. **77.** $(17+a)(3-a)$.
78. $(1+m+p)(1-m-p)$. **79.** $(bx-a)(ax-b)$.
80. $(3b-c+4)(3b-c-4)$. **81.** $(c+1)(c^2-c+1)(x+1)(x-1)$.
82. $(3x-b)(x+2a)$. **83.** $(m-n)(m+n+x)(m+n-x)$.
84. $(a+b)(c+a-b)(c-a+b)$.
85. $(x+2)(x^2-2x+4)(x^2+1)(x+1)(x-1)$. **86.** $(x+1)(x+7)(2x-3)$.
87. $(2x+5y)(x-3y)(2x-5y)$. **88.** $325a^3b^3(x^2-a^2)^2(x+2a)$.
89. $2x^2-9x+9$. **90.** $2x^3(x^2-4)(x^2-16)$.
91. H.C.F. $=a+b+c$, L.C.M. $=(a+b+c)(a-b)(b-c)(c-a)$.
92. $a+b-c$. **93.** $(a-b)^2(a+b)$. **95.** $(a^4-b^4)(a+b-2c)$.
97. H.C.F. $=(x-7)(x-3)$,
L.C.M. $=(x-1)(x-2)(x-3)(x-4)(x-5)(x-7)$.
98. $\dfrac{1}{(1-x)^2}$. **99.** $\dfrac{x-9}{(x^2-9)(x-3)}$. **100.** $\dfrac{6x+1}{(2x+1)^2(2x-1)}$
101. $\dfrac{2}{x}$. **102.** $\dfrac{4}{(1-x^2)^2}$. **103.** 1. **104.** 0. **105.** $\dfrac{x}{9}$.
106. $\dfrac{2x-y}{x^2-y^2}$. **107.** $y-x$. **108.** ab. **109.** $2(ac+bd)(ad+bc)$.
110. 1. **111.** 1. **112.** $\dfrac{(x^2+2)(x^4+1)}{x}$. **113.** $\dfrac{1}{f-g}$.
114. $\dfrac{1}{x+1}$. **115.** $x(1+x-x^2)$. **116.** $\dfrac{3abc}{a+b}$. **117.** $a+b$. **118.** 1.

XXIII. a. Page 180.

1. 6.	**2.** $1\frac{3}{10}$.	**3.** $\frac{1}{5}$.	**4.** 1.	**5.** 20.
6. 2.	**7.** $\frac{7}{17}$.	**8.** 0.	**9.** 2.	**10.** $-6\frac{5}{6}$.
11. 5.	**12.** 6.	**13.** $-\frac{8}{11}$.	**14.** $-\frac{7}{18}$.	**15.** 1.
16. -10.	**17.** -4.	**18.** $3\frac{3}{8}$.	**19.** 3.	**20.** 4.
21. 6.	**22.** 13.	**23.** -7.	**24.** 2.	**25.** $2\frac{1}{2}$.
26. 4.	**27.** $1\frac{1}{2}$.	**28.** 14.	**29.** $\frac{1}{4}$.	**30.** $2\frac{1}{4}$.
31. $\frac{1}{6}$.	**32.** 3.	**33.** 20.		

XXIII. b. Page 182.

1. $\frac{2b-3a}{a-5b}$.	**2.** $a+b$.	**3.** $\frac{b^2-a^2}{2b}$.	**4.** $\frac{a^2-ab+b^2}{a-b}$.
5. 3.	**6.** $m-n$.	**7.** $-\frac{ab}{a+b+c}$.	**8.** $\frac{a^2-2ab+bc}{c-b}$.
9. a.	**10.** $\frac{7bc}{9b+4c}$.	**11.** $\frac{2ab}{a+b}$.	**12.** $17a$.
13. $\frac{1}{c}$.	**14.** $3a+2b$.	**15.** $\frac{a+b}{2}$.	**16.** $\frac{a^2-2b^2}{3a-4b}$.
17. $\frac{a}{17}$.	**18.** $\frac{a^2}{b}$.	**19.** $\frac{bc^2}{a^2}$.	**20.** a.

21. $a+b$. **22.** $\frac{2a}{21}$. **23.** $\frac{a+2b}{2}$. **24.** $\frac{a}{3}$. **25.** $\frac{b(2a-b)}{a}$.

XXIII. c. Page 185.

1. $x=\frac{al-bm}{a^2-b^2},\ y=\frac{am-bl}{a^2-b^2}$.

2. $x=\frac{nq-mr}{lq-mp},\ y=\frac{lr-np}{lq-mp}$.

3. $x=\frac{bc}{a^2+b^2},\ y=\frac{ac}{a^2+b^2}$.

4. $x=\frac{a^2+ab+b^2}{a+b},\ y=-\frac{ab}{a+b}$.

5. $x=\frac{a'^2-a}{a'-a^2},\ y=\frac{1-aa'}{a'-a^2}$.

6. $x=\frac{q^2-pr}{qr-p^2},\ y=\frac{pq-r^2}{qr-p^2}$.

7. $x=\frac{a+a'}{a'b+ab'},\ y=\frac{b'-b}{a'b+ab'}$

8. $x=2a,\ y=2b$.

9. $x=\frac{2}{3}a,\ y=\frac{1}{2}b$.

10. $x=\frac{pa}{q},\ y=\frac{rb}{p}$.

11. $x=\dfrac{mm'(m+m')}{m^2+m'^2}$, $y=\dfrac{mm'(m-m')}{m^2+m'^2}$.

12. $x=\dfrac{qn}{ql-pm}$, $y=\dfrac{pn}{mp-lq}$. 13. $x=\dfrac{c(a+b)}{2a}$, $y=\dfrac{c(a-b)}{2a}$.

14. $x=a+b$, $y=a-b$. 15. $x=3a$, $y=-2b$.

16. $x=\dfrac{2aa'b}{ab'+a'b}$, $y=\dfrac{2abb'}{ab'+a'b}$. 17. $x=a$, $y=0$.

18. $x=m+l$, $y=m+l$. 19. $x=\dfrac{a}{b}$, $y=\dfrac{b}{c}$.

20. $x=a+b$, $y=a-b$. 21. $x=a^3-b^3$, $y=a^3+b^3$.

XXIV. Page 188.

1. 40. 2. 60. 3. 55. 4. 22.
5. 30. 6. 54. 7. 42. 8. 48, 23.
9. $21\frac{9}{11}'$ past one. 10. $17\frac{5}{11}'$ past three.
11. $32\frac{8}{11}'$ past six. 12. $5\frac{10}{11}'$ past two. 13. 378, 216.
14. 15 persons ; 5 dollars. 15. 8 yards at \$4.50 ; 16 yards at \$4.
16. 17, 15. 17. 3 miles per hour.
18. 54. 19. $2\frac{1}{2}$ miles per hour.
20. $21\frac{9}{11}'$ and $54\frac{6}{11}'$ past seven At $5\frac{5}{11}'$ past. 21. $\frac{8}{12}$.
22. 10 p.m. ; halfway. 23. $1\frac{1}{3}$ hours. 24. \$200.
25. 30 miles. 26. \$36000. 27. \$200.
28. 4 and 3 gallons. 29. $\frac{3}{5}$ and $\frac{2}{5}$ of a pint. 30. $\frac{pa}{p+q}$ miles.
31. 111 and 126 miles. 32. Coffee to chicory as 7 to 2.
33. $c-b$ and $a-c$ lbs. 34. $\frac{20c}{a}$, $\frac{80c}{b}$ yards. 36. 60 miles.

XXV. a. Page 194.

1. ± 5. 2. ± 4. 3. 3, -25. 4. 1, -25.
5. 3, 7. 6. ± 8. 7. 3, -6. 8. 2, -7.
9. 9, -4. 10. 9, -8. 11. 31, -11. 12. 20, -11.
13. 4, -17. 14. 13, -12. 15. 11, -17. 16. 8, 15.
17. 7, 6. 18. 23, -1. 19. 6, $-\frac{16}{3}$. 20. $\frac{1}{3}$, $-\frac{3}{5}$.
21. $\frac{3}{2}$, $-\frac{1}{3}$. 22. $\frac{1}{5}$, -4. 23. ± 9.

XXV. b. Page 197.

1. $\frac{11}{5}, -5.$	**2.** $11, \frac{11}{3}.$	**3.** $3, \frac{7}{6}.$	**4.** $\frac{15}{8}, -2.$
5. $\frac{7}{3}, 5.$	**6.** $2, -\frac{11}{6}.$	**7.** $\frac{3}{4}, -5.$	**8.** $\frac{7}{2}, -3.$
9. $\frac{13}{3}, -\frac{11}{3}.$	**10.** $\frac{7}{4}, \frac{2}{3}.$	**11.** $\frac{3}{4}, -\frac{4}{5}.$	**12.** $\frac{5}{8}, -3.$
13. $-\frac{5}{7}, -\frac{1}{3}.$	**14.** $\frac{9}{10}, -\frac{3}{5}.$	**15.** $\frac{13}{6}, -\frac{2}{3}.$	**16.** $3, -\frac{7}{5}.$
17. $\frac{a}{3}, -\frac{a}{5}.$	**18.** $\frac{3a}{7}, -\frac{a}{3}.$	**19.** $\frac{7k}{3}, -\frac{k}{2}.$	**20.** $-\frac{5k}{4}, -\frac{2k}{3}.$
21. $\frac{4c}{3}, -\frac{5c}{4}.$	**22.** $3, -\frac{4}{3}.$	**23.** $5, -\frac{5}{2}.$	**24.** $4, \frac{7}{5}.$
25. $3, -1.$	**26.** $2, \frac{1}{3}.$	**27.** $4, \frac{11}{2}.$	**28.** $7, 2.$
29. $11, 2.$	**30.** $4, \frac{4}{3}.$	**31.** $13, \frac{2}{3}.$	**32.** $6, \frac{40}{13}.$
33. $2, \frac{39}{8}.$	**34.** $3, -\frac{1}{2}.$	**35.** $12, -2.$	**36.** $5, \frac{23}{7}.$
37. $3a, \frac{3a}{2}.$	**38.** $2c, \frac{11c}{14}.$	**39.** $a, \frac{ab}{a-2b}.$	

XXV. c. Page 201.

1. $\frac{5}{3}, -3.$	**2.** $\frac{3}{2}, -5.$	**3.** $1, \frac{7}{2}.$	**4.** $\frac{3 \pm \sqrt{29}}{2}.$
5. $-4, -\frac{1}{5}.$	**6.** $\frac{7 \pm \sqrt{5}}{2}.$	**7.** $1, -\frac{7}{8}.$	**8.** $\frac{17 \pm \sqrt{89}}{10}.$
9. $7, -\frac{5}{2}.$	**10.** $\frac{1 \pm \sqrt{13}}{6}.$	**11.** $\frac{1}{3}, -2.$	**12.** $3, -\frac{11}{2}.$
13. $\frac{7}{6}, -1.$	**14.** $\frac{7}{4}, \frac{3}{2}.$	**15.** $\frac{7}{11}, -3.$	**16.** $-\frac{6}{5}, -4.$
17. $\frac{9}{10}, -\frac{5}{6}.$	**18.** $\frac{8}{3}, -\frac{3}{4}.$	**19.** $\frac{2}{7}, -14.$	**20.** $\frac{5}{12}, -\frac{3}{8}.$
21. $\frac{3}{5}, -\frac{2}{5}.$	**22.** $\frac{5}{2}, -\frac{7}{2}.$	**23.** $\frac{9a}{4}, -\frac{4a}{3}.$	**24.** $\frac{9a}{4}, \frac{4a}{3}.$
25. $\frac{5b}{3}, -\frac{7b}{3}.$	**26.** $\frac{7b}{6}, -\frac{5b}{6}.$	**27.** $2a, 2b.$	**28.** $2a, -8.$

29. $0, \frac{2a+b}{3}$. **30.** $0, \frac{b-2}{a}$. **31.** $\pm 2, \pm 1$. **32.** $\pm 2, \pm 3$.

33. $1, -2$. **34.** $3, -2$. **35.** $\pm 4, \pm \frac{1}{4}$. **36.** $\pm a, \pm b$.

37. $2, -3$. **38.** $\pm 3, \pm 4$. **39.** $3, -2, 4, -3$. **40.** $4a, -2a, a$.

XXV. d. Page 201 B.

1. $1, -1, -1$. **2.** $1, -1, 2$. **3.** $1, 2, -2$. **4.** $1, -3, -5$.

5. $2, -1, -1$. **6.** $0, 1, 1, -2$. **7.** $3, 2, -5$. **8.** $5, 2, -7$.

9. $7, -3, -4$. **10.** $-2a, -2a, 4a$. **11.** $0, 6a, 6a, -12a$.

12. $1·05, -3·05$. **13.** $3·90, -·90$. **14.** $·66, -1·66$.

15. $18·55, 17·45$. **16.** $5·99, 1·01$. **17.** $3·18, 2·32$.

18. $·55, -·22$. **19.** $1·4, ·6$.

20. $\frac{a}{2}(\sqrt{5}-1), -\frac{a}{2}(\sqrt{5}+1)$. $7·416, -19·416$.

21. $\frac{1}{2}(a \pm \sqrt{a^2-4c^2})$. $13·292, 2·708$.

XXVI. a. Page 203.

1. $x=17, 11$; $y=11, 17$. **2.** $x=37, 14$; $y=14, 37$.

3. $x=53, 21$; $y=21, 53$. **4.** $x=14, -9$; $y=9, -14$.

5. $x=27, -19$; $y=19, -27$. **6.** $x=43, -25$; $y=25, -43$.

7. $x=71, 13$; $y=13, 71$. **8.** $x=33, -41$; $y=41, -33$.

9. $x=52, -74$; $y=74, -52$. **10.** $x=43, -51$; $y=-51, 43$.

11. $x=29, -47$; $y=47, -29$. **12.** $x=22, -87$; $y=-87, 22$.

13. $x=\pm 8, \pm 5$; $y=\pm 5, \pm 8$. **14.** $x=\pm 13, \pm 1$; $y=\pm 1, \pm 13$.

15. $x=\pm 4, \pm 7$; $y=\pm 7, \pm 4$. **16.** $x=13, 3$; $y=3, 13$.

17. $x=10, 5$; $y=5, 10$. **18.** $x=9, -5$; $y=5, -9$.

19. $x=12, -6$; $y=6, -12$. **20.** $x=11, -8$; $y=8, -11$.

21. $x=9, 4$; $y=4, 9$. **22.** $x=5, 4$; $y=4, 5$.

23. $x=7, -4$; $y=4, -7$. **24.** $x=10, 4$; $y=4, 10$.

25. $x=12, -2$; $y=2, -12$. **26.** $x=1$; $y=1$.

27. $x=4, 3$; $y=3, 4$. **28.** $x=\frac{1}{a}$; $y=\frac{1}{b}$.

29. $x=\pm 1$; $y=\pm 1$.

XXVI. b. PAGE 205.

1. $x=7,\ 4$; $y=4,\ 7$. **2.** $x=8,\ 5$; $y=5,\ 8$.

3. $x=14,\ 9$; $y=9,\ 14$. **4.** $x=7,\ -5$; $y=5,\ -7$.

5. $x=11,\ -7$; $y=7,\ -11$. **6.** $x=13,\ 0$; $y=0,\ -13$.

7. $x=\pm 6,\ \pm 4$; $y=\pm 4,\ \pm 6$. **8.** $x=\pm 7,\ \pm 3$; $y=\pm 3,\ \pm 7$.

9. $x=\pm 9,\ \pm 5$; $y=\pm 5,\ \pm 9$. **10.** $x=\pm 9,\ \pm 3$; $y=\pm 3,\ \pm 9$.

11. $x=\frac{6}{5},\ \frac{8}{3}$; $y=\frac{8}{3},\ \frac{6}{5}$. **12.** $x=\pm 6,\ \pm 5$; $y=\pm 5,\ \pm 6$.

13. $x=4,\ 2$; $y=2,\ 4$. **14.** $x=7,\ -3$; $y=3,\ -7$.

15. $x=5,\ 3$; $y=3,\ 5$. **16.** $x=4,\ -2$; $y=2,\ -4$.

17. $x=8,\ -2$; $y=2,\ -8$. **18.** $x=5,\ 1$; $y=1,\ 5$.

19. $x=5,\ 1$; $y=1,\ 5$; **20.** $x=\frac{1}{6},\ -\frac{1}{5}$; $y=\frac{1}{5},\ -\frac{1}{6}$.

XXVI. c. PAGE 208.

1. $x=4,\ -\frac{3}{5}$; $y=3,\ -20$. **2.** $x=\pm 3$; $y=\pm 2$.

3. $x=12,\ 8$; $y=2,\ -2$. **4.** $x=2,\ \frac{10}{3}$; $y=5,\ 3$.

5. $x=4,\ 7$; $y=1,\ 10$. **6.** $x=4,\ -3$; $y=1,\ -\frac{4}{3}$.

7. $x=1,\ -\frac{71}{17}$; $y=4,\ \frac{112}{17}$. **8.** $x=\pm 2,\ \pm\frac{4}{\sqrt{5}}$; $y=\pm 1,\ \pm\frac{3}{\sqrt{5}}$.

9. $x=2,\ \frac{5}{8}$; $y=-7,\ -\frac{1}{8}$. **10.** $x=\pm 4,\ \pm 6$; $y=\pm 2,\ \pm 4$.

11. $x=\pm 3,\ \pm 4$; $y=\pm 2,\ \pm 5$. **12.** $x=\pm\frac{3}{2},\ \pm\frac{1}{2}$; $y=\pm\frac{1}{2},\ \pm\frac{3}{2}$.

13. $x=\pm 2,\ \pm 1$; $y=\pm 1,\ \pm 2$. **14.** $x=\pm 2,\ \pm 5$; $y=\pm 3,\ \pm 6$.

15. $x=\pm 7,\ \pm\sqrt{3}$; $y=\pm 2,\ \mp 3\sqrt{3}$.

16. $x=\pm 3,\ \pm 36$; $y=\pm 5,\ \mp\frac{23}{2}$. **17.** $x=5,\ 3$; $y=3,\ 5$.

18. $x=7,\ -6$; $y=6,\ -7$. **19.** $x=6,\ -2$; $y=2,\ -6$.

20. $x=7,\ 1,\ 4\pm\sqrt{28}$; $y=1,\ 7,\ 4\mp\sqrt{28}$.

21. $x=4,\ 3,\ 6,\ 2$; $y=\frac{3}{2},\ 2,\ 1,\ 3$.

22. $x=2,\ \frac{2}{3},\ 4,\ \frac{1}{3}$; $y=2,\ 6,\ 1,\ 12$.

XXVII. PAGE 211.

1. 13. **2.** 45, 9. **3.** 7, 8. **4.** 3. **5.** 15, 12.
6. 9. **7.** 7 hours. **8.** 7, 5. **9.** 90 yards, 160 yards.
10. 55 feet, 30 feet. **11.** 36′, 60′. **12.** 6.
13. 5 dollars. **14.** 20. **15.** 18 cents. **16.** 3 feet.
17. 4 inches. **18.** 121 square feet. **19.** 4 cents.
20. 40, 12; 30, 16 yards. **21.** 56. **22.** 50. **23.** 25.
24. $6\frac{2}{3}$ miles. **25.** 75. **26.** 20, 30 miles an hour.
27. 40 and 45 miles an hour. **28.** 10 gallons.
29. A, 16; B, 14. **30.** Distance, 12 miles; rate, 8 miles an hour.
31. $\frac{a}{2}(-1 \pm \sqrt{5})$. **32.** 3·7 cm., 2·3 cm.
33. $AP = 20{\cdot}9$ cm., $BP = 12{\cdot}9$ cm. **35.** 8·4 cm.
36. 2·6 cm., 1·6 cm. **37.** 9 cm., 4 cm.
39. (i) 3, 4; (ii) 5, 6; (iii) 5·2, 0·8; (iv) 5·7, 2·3.

XXVIII. a. PAGE 216.

1. $(x^2+4x+16)(x^2-4x+16)$. **2.** $(9a^2+3ab+b^2)(9a^2-3ab+b^2)$.
3. $(x^2+3xy+y^2)(x^2-3xy+y^2)$. **4.** $(m^2+4mn-n^2)(m^2-4mn-n^2)$.
5. $(x^2+2xy-y^2)(x^2-2xy-y^2)$.
6. $(2x^2+9xy-3y^2)(2x^2-9xy-3y^2)$.
7. $(2m^2+6mn+3n^2)(2m^2-6mn+3n^2)$.
8. $(3x^2+xy+2y^2)(3x^2-xy+2y^2)$.
9. $(x^2+3xy-5y^2)(x^2-3xy-5y^2)$.
10. $(4a^2-6ab+b^2)(4a^2+6ab+b^2)$.
11. $\left(\frac{3}{ab}-1\right)\left(\frac{9}{a^2b^2}+\frac{3}{ab}+1\right)$. **12.** $\left(6a-\frac{b}{2}\right)\left(36a^2+3ab+\frac{b^2}{4}\right)$.
13. $\left(\frac{x}{5}+y\right)\left(\frac{x^2}{25}-\frac{xy}{5}+y^2\right)$. **14.** $\left(\frac{mn}{9}-1\right)\left(\frac{m^2n^2}{81}+\frac{mn}{9}+1\right)$.

15. $\left(\frac{ab}{5}+10\right)\left(\frac{a^2b^2}{25}-2ab+100\right)$.

16. $\left(\frac{x}{8}-\frac{4}{x}\right)\left(\frac{x^2}{64}+\frac{1}{2}+\frac{16}{x^2}\right)$.

17. $(y-3x)(x+y)(x-y)$.

18. $(m-5n)(2n+3m)(2n-3m)$.

19. $(ax+b)(bx+a)$.

20. $(x^2z^2+y^2)(xy+z)(xy-z)$.

21. $(a^2+bx)(a+x)$.

22. $(mn-p)(pm-n)$.

23. $(3ab-2x)(2ax-3b)$.

24. $(2x+3y)(a^2+xy)$.

25. $(2x-3y)(a^2+xy)$.

26. $\{ax+(a+1)\}\{(a-1)x+a\}$.

27. $(x-a)(3x-a-2b)$.

28. $\{ax+2(b-c)y\}\{2ax-(3b-4c)y\}$.

29. $\{(a-1)x+a\}\{(a-2)x+(a-1)\}$.

30. $\{(a+1)x-(b-1)y\}(ax+by)$.

31. $(b+c-1)(b^2+c^2+1-bc+c+b)$.

32. $(a+2c+1)(a^2+4c^2+1-2ac-a-2c)$.

33. $(a+b+2c)(a^2+b^2+4c^2-ab-2bc-2ca)$.

34. $(a-3b+c)(a^2+9b^2+c^2+3ab+3bc-ca)$.

35. $(a-b-c)(a^2+b^2+c^2+ab-bc+ca)$.

36. $(2a+3b+c)(4a^2+9b^2+c^2-6ab-3bc-2ca)$.

37. $(x^4-9x^2+81)(x^2+3x+9)(x^2-3x+9)$.

38. $(a^4-4a^2b^2-b^4)(a^2+b^2)(a+b)(a-b)$.

39. $(a+b+c-d)(a+b-c+d)(c+d+a-b)(c+d-a+b)$.

40. $\left(x^4+\frac{1}{16}\right)\left(x^2+\frac{1}{4}\right)\left(x+\frac{1}{2}\right)\left(x-\frac{1}{2}\right)$.

41. $(x^8+y^8)(x^4+y^4)(x^2+y^2)(x+y)(x-y)$.

42. $(x^6+x^3y^3+y^6)(x^6-x^3y^3+y^6)(x^2+xy+y^2)(x^2-xy+y^2)(x+y)(x-y)$.

43. $\left(\frac{1}{x}+1\right)\left(\frac{1}{x}-1\right)(a-2x)(a^2+2ax+4x^2)$.

44. $(x^2+y^2)(x^4-x^2y^2+y^4)(x-2y)(x^2+2xy+4y^2)$.

45. $(x^2+4)(x^4-4x^2+16)(x+1)(x^2-x+1)$.

46. $\left(\frac{2}{a}+\frac{3}{b}\right)\left(\frac{2}{a}-\frac{3}{b}\right)(a+b)(a^2-ab+b^2)$.

47. $\left(\frac{1}{3x}+\frac{y}{2}\right)\left(\frac{1}{3x}-\frac{y}{2}\right)\left(\frac{xy}{2}-1\right)\left(\frac{x^2y^2}{4}+\frac{xy}{2}+1\right)$.

48. $(x^2+5)(x^2-5)\left(x+\frac{1}{2}\right)\left(x-\frac{1}{2}\right)$.

49. $(x+1)(x^2-x+1)(x^2+4)(x+2)(x-2)$.

50. $(x-1)(x^2+x+1)(4x^2+9)(2x+3)(2x-3)$.

XXVIII. b. Page 220.

1. $4x^2-49y^2+42yz-9z^2$. 2. $9x^4+26x^2y^2+49y^4$.
3. $25x^4-115x^2y^2+81y^4$. 4. $49x^4-64x^2y^2+48xy^3-9y^4$.
5. x^6-y^6. 6. $(x+y)^4+4(x+y)^2+16$. 7. $16x^2(1-4x^2)$.
8. $48a^2(a^4-1)$. 9. x^6-64. 10. x^6-729a^6.
11. $\frac{a^4}{x^2}-3a^2-x^2-\frac{x^4}{a^2}$. 12. $64x^4(9x^2-1)$. 13. $x^8+a^4x^4+a^8$.
14. $1+x^8+x^{16}$. 15. $a^{12}-3a^8x^4+3a^4x^8-x^{12}$.
16. $1-2x^8+x^{16}$. 17. $x^6-14x^4+49x^2-36$.
18. $x^6-14x^4+49x^2-36$. 19. x^6-64. 20. $a^4-18a^2b^2+81b^4$.
21. $a^3+b^3+c^3-3abc$. 22. $7x+y+z$.
23. $(x^4-4a^2x^2+16a^4)(x^2-2ax+4a^2)$. 24. $5x+7y-6z$.
25. $x+5$. 26. $2x(x+1)$. 27. $5(x-13)$.
28. $(x+3)(x^2+2x+4)$. 29. $(7x-3)(x-1)$. 30. $a-b$.
31. $x^3-3x^2y-3xy^2+y^3$. 32. $x^4-4x^2yz+7y^2z^2$.
33. $1+9x^2+4y^2+6xy+2y-3x$. 34. $(x+1)(x-3)$.
35. $(2a-5)(2a-7)$. 36. $(x-a)(x-b)$.
37. $a^2+9x^2+4y^2-6xy+2ay+3ax$.
38. $9+4x^2+16y^2-8xy+12y+6x$.
45. $\frac{m(m^2+3n^2)}{4}$. 47. $(3a^2+b^2)(a^2+3b^2)$.
49. $\frac{1}{16}(9p^2-5q^2)(9q^2-5p^2)$. 50. $16ab^3$.

XXIX. a. Page 223.

1. $x+c$. 2. x^2-ax+b. 3. $x^2+2bx-ax-2ab$.
4. $x^2-(p+q)x+2q(p-q)$. 5. $x^2-(m+n)x+m(m-n)$.
6. $ax+a+1$. 7. x^2+bx+a^2. 8. $2lx-(3m-4n)y$.
9. $(a+2)x+(a+1)y$. 10. $x+b$.
11. $(x+1)^6+3(x+1)^4+3(x+1)^2+1$.
12. $(m+1)b^2x^2+(n+1)(m+1)abx+(n+1)a^2$.

13. $(m-1)x+m.$ **14.** $mx-n.$ **15.** $ap-bq.$
16. $ax+b.$ **17.** $2ax-3.$ **18.** $x+2ab.$
19. $(x^2-1)(x^2-px+q)(x^2-qx+p).$
20. $\{px-(p-1)\}\{(p+1)x+p\}\{(p+2)x+p+1\}.$
21. $\{(a-3)x+a+1\}\{(a-2)x-a\}\{ax-(a+4)\}.$

XXIX. b. Page 227.

1. $x-7.$ **2.** $2-\frac{1}{m}.$ **3.** $a+2x.$
4. $1+x-x^2.$ **5.** $1+2x-x^2.$ **6.** $1+x.$
7. $x-2a.$ **8.** $a-3x.$ **9.** $x-y.$
10. $x^2+(p-1)x-1.$ **11.** $1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}.$ **12.** $1-x-\frac{x^2}{2}-\frac{x^3}{2}.$
13. $2+\frac{x}{2}-\frac{x^2}{16}+\frac{x^3}{64}.$ **14.** $1-\frac{x}{2}-\frac{5x^2}{8}-\frac{5x^3}{16}.$
15. $a-\frac{x}{2a}-\frac{x^2}{8a^3}-\frac{x^3}{16a^5}.$ **16.** $x+\frac{a^2}{2x}-\frac{a^4}{8x^3}+\frac{a^6}{16x^5}.$
17. $a^2-\frac{3x^2}{2a^2}-\frac{9x^4}{8a^6}-\frac{27x^6}{16a^{10}}.$ **18.** $3a+2x-\frac{2x^2}{3a}+\frac{4x^3}{9a^2}.$
19. $x-\frac{a^3}{3x^2}-\frac{a^6}{9x^5}.$ **20.** $2+\frac{x}{12}-\frac{x^2}{288}.$
21. $\frac{1}{a}+3a^2x-9a^5x^2.$ **22.** $1-2x+3x^2.$
23. $3x^2-x-1.$ **24.** $4-x-\frac{x^2}{16}.$

XXIX. c. Page 230.

19. 0. **20.** 0. **26.** 0.

XXIX. d. Page 232.

1. 0. **2.** 1. **3.** 1. **4.** $a+b+c.$
5. 1. **6.** $\frac{1}{abc}.$ **7.** $\frac{1}{abc}.$ **8.** 1.
9. $d.$ **10.** $\frac{1}{(x-a)(x-b)(x-c)}.$ **11.** $\frac{x^2}{(x+a)(x+b)(x+c)}.$
12. $(a+b+c)^2.$ **13.** $-\frac{a+b+c}{3}.$ **14.** $\frac{1}{3}.$ **15.** $a+b+c.$
16. $bc+ca+ab.$ **17.** $abc.$ **18.** $(b+c)(c+a)(a+b).$

XXIX. e. Page 238.

1. 5. **2.** 10. **3.** $\frac{b}{a}$. **4.** $\frac{p}{16q}$.

5. $\frac{5c}{b}$. **6.** $\frac{d-a^4}{2a^3-c}$. **7.** $a=c,\ b=\frac{a^2}{4}+2$. **8.** 6.

9. $\pm 3a$. **10.** $\pm\sqrt{\frac{2n}{m}}$. **11.** $b^3=27c^2$.

12. $c=a(b-a^2)^2,\ d=(b-a^2)^3$, whence $c^3=a^3d^2$. **13.** 32.

15. $(x-1)(x-2)(x-3)$. **16.** $(x+2)(x-3)(x-4)$.

17. $(x+2)(x+3)(x+4)$. **18.** $(x-3)(x-5)(x+7)$.

19. $(x-2)(x-5)(x+7)$. **20.** $(x+1)(x+2)(x-11)$.

21. $(x+1)(3x+2)(2x-1)$. **22.** $(x+2)(3x-1)(2x-3)$.

23. $x^6-x^5y+x^4y^2-x^3y^3+x^2y^4-xy^5+y^6$.

24. $x^7-x^6y+x^5y^2-x^4y^3+x^3y^4-x^2y^5+xy^6-y^7$.

25. $x^5+x^4y+x^3y^2+x^2y^3+xy^4+y^5$.

26. $x^8+x^7y+x^6y^2+x^5y^3+x^4y^4+x^3y^5+x^2y^6+xy^7+y^8$.

27. $x^2+(a-2)x+a$. **28.** $(a+1)x^2+ax+a-3$. **29.** 6 or $\frac{2}{3}$.

30. 13. **34.** 3005. **35.** $-37a^3$.

XXX. a. Page 244.

1. $\frac{2}{x^{\frac{1}{4}}}$. **2.** $\frac{3}{a^{\frac{2}{3}}}$. **3.** $\frac{4a^3}{x^2}$. **4.** $3a^2$. **5.** $\frac{a^2}{4}$.

6. $\frac{x^{\frac{1}{2}}}{5}$. **7.** $\frac{3c^4x^2}{5a^3y^2}$. **8.** $\frac{x^a b^a}{y^b}$. **9.** $\frac{6}{x^{\frac{1}{2}}}$. **10.** $\frac{a^{\frac{1}{2}}}{2}$.

11. y^2. **12.** $\frac{1}{3a^2x^2}$. **13.** $\frac{1}{x^{\frac{3}{2}}}$. **14.** $\frac{x^{\frac{3}{5}}}{4}$. **15.** $2y^{\frac{3}{2}}$.

16. $x^{\frac{5}{4}}$. **17.** $\frac{a}{x^{\frac{1}{2}}}$. **18.** $\frac{1}{a^{\frac{2}{3}}}$. **19.** $\frac{1}{a^2}$. **20.** $\sqrt[5]{x^3}$.

21. $\frac{1}{\sqrt{a}}$. **22.** $\frac{5}{\sqrt{x}}$. **23.** $\frac{2}{\sqrt[x]{a}}$. **24.** $\frac{1}{2\sqrt[3]{a}}$. **25.** $2\sqrt[4]{b^3}$.

26. $\frac{1}{2\sqrt[3]{c}}$. **27.** $\sqrt[z]{x}$. **28.** $\frac{2}{\sqrt[6]{a^5}}$. **29.** $\frac{\sqrt{a}}{2\sqrt[3]{x^2}}$. **30.** $\frac{21}{\sqrt{a^3}}$.

31. $\frac{2}{\sqrt{a}}$. **32.** $\frac{1}{3\sqrt{a^3}}$. **33.** $\frac{4}{\sqrt[3]{x^2}}$. **34.** $\frac{1}{\sqrt[3]{x^{a+2}}}$. **35.** $\sqrt[6]{a^{13}}$.

36. $\sqrt[5]{a^x}$. **37.** $\sqrt[2a]{x^5}$. **38.** $\frac{1}{\sqrt[2a]{x}}$. **39.** $\frac{1}{\sqrt[x]{a}}$. **40.** $\sqrt[6]{a^n}$.

41. 8. **42.** $\frac{1}{32}$. **43.** 25. **44.** $\frac{1}{4}$. **45.** $\frac{1}{216}$.

46. 625. **47.** 9. **48.** $\frac{3}{2}$. **49.** $\frac{27}{8}$. **50.** $\frac{2187}{128}$.

XXX. b. Page 247.

1. a^6b^9. **2.** $\frac{x^{\frac{4}{3}}}{y}$. **3.** $\frac{1}{y^{2a+3b}}$. **4.** $\frac{1}{2x^{\frac{1}{2}}y^{\frac{1}{2}}}$. **5.** $\frac{4}{9a^2x^2}$.

6. $16ac^4$. **7.** $\frac{x^{\frac{1}{3}}}{y^{\frac{1}{4}}}$. **8.** $x^{\frac{1}{6}}$. **9.** $\frac{3ax}{2}$. **10.** x^{n-1}.

11. $\frac{1}{x^{n+1}}$. **12.** $\frac{1}{x^{\frac{1}{a}}}$. **13.** $\frac{1}{a^{\frac{2}{3}}b^{\frac{1}{2}}}$. **14.** $a^{\frac{1}{2}}$. **15.** x^{b+1}.

16. $\frac{1}{x^{\frac{1}{2}}}$. **17.** $\frac{1}{x^2}$. **18.** ab^2. **19.** $a+b$. **20.** $\frac{1}{(x^2-y^2)^{3n}}$.

21. $\frac{1}{a^5}$. **22.** $b^{\frac{3}{2}}$. **23.** $x^{\frac{1}{9}}$. **24.** $\frac{a+b}{(a-b)^{\frac{1}{2}}}$. **25.** $c^{\frac{7}{2}}$.

26. $\frac{x^2}{a^3}$. **27.** $\frac{1}{a^{\frac{8}{3}}}$. **28.** $ab(b^6-a^6)^{\frac{1}{3}}$. **29.** $a^{n(n-1)}+a$.

30. $x^{n(n-1)}+x^{n-1}$. **31.** $a^{4n(p-q)}$. **32.** x^b. **33.** $\frac{x^7}{y^7}$.

34. $x^{\frac{1}{7}}y^{\frac{35}{6}}$. **35** 2^{n^2}. **36.** $\frac{1}{4}$. **37.** 4. **38.** 1

XXX. c. Page 250.

1. $12x^{\frac{2}{3}}-20x^{\frac{1}{3}}+41-15x^{-\frac{1}{3}}+24x^{-\frac{2}{3}}$.

2. $9a^{\frac{4}{5}}-9a^{\frac{2}{5}}-25+23a^{-\frac{2}{5}}+6a^{-\frac{4}{5}}$.

3. $2c^{2x}-9c^x-34+31c^{-x}-6c^{-2x}$. **4.** $8x^{3a}+14x^a-3x^{-a}-9x^{-3a}$.

5. $7x^{\frac{2}{3}}-2x^{\frac{1}{3}}+1$. **6.** $3a^{\frac{1}{3}}-3a^{-\frac{1}{3}}+2a^{-1}$.

7. $8a^{-2}+7a^{-1}+6$. **8.** $5b^{\frac{1}{2}}+4b^{\frac{1}{6}}+3b^{-\frac{1}{6}}+2b^{-\frac{1}{2}}$.

9. $7a^{2x}+3a^x-4$. **10.** $c^{2n}-1+c^{-2n}$.

11. $3x^{\frac{1}{2}}-2+x^{-\frac{1}{2}}$. **12.** $5a^{\frac{2}{3}}-3a^{\frac{1}{3}}+4$.

13. $2x^{\frac{n}{2}}-4+3x^{-\frac{n}{2}}$.
14. $a^{2x}-3a^{x}-2$.
15. $a^2+2a-16a^{-2}-32a^{-3}$.
16. $1-x^{\frac{1}{6}}-2x^{\frac{1}{3}}+2x^{\frac{2}{3}}$.
17. $4a^{\frac{8}{3}}-8a^{\frac{4}{3}}-5+10a^{-\frac{4}{3}}+3a^{-\frac{8}{3}}$.
18. $x^{\frac{1}{2}}-2x^{\frac{1}{6}}+4x^{-\frac{1}{6}}-8x^{-\frac{1}{2}}$.
19. $1-2a-2a^{\frac{3}{2}}$.
20. $2x^{\frac{1}{4}}-3x^{-\frac{1}{12}}-x^{-\frac{5}{12}}$.
21. $3x^{-2}-3x^{-1}y^{\frac{1}{2}}+y$.
22. $2x^{\frac{3}{4}}-3y^{\frac{1}{4}}+4x^{-\frac{3}{4}}y^{\frac{1}{2}}$.
23. $9x^{\frac{2}{3}}y^{-1}+2x^{\frac{1}{3}}y^{-\frac{1}{2}}-9$.
24. $\frac{1}{4}x^{-1}+1-3y^{\frac{1}{3}}$.

XXX. d. Page 252.

1. $x-4x^{\frac{1}{2}}-21$.
2. $16x^2-8-15x^{-2}$.
3. $49x^2-81y^{-2}$.
4. $x^{m}y^{-n}-x^{-m}y^{n}$.
5. $a^{2x}-4+4a^{-2x}$.
6. $a^{2x}+2a^{x+\frac{1}{x}}+a^{\frac{2}{x}}$.
7. $x^{a}-x^{-\frac{a}{2}}+\frac{1}{4}x^{-2a}$.
8. $20x^{2a}y^{2b}+13-15x^{-2a}y^{-2b}$.
9. $\frac{1}{9}a^{\frac{2}{3}}-\frac{2}{3}+a^{-\frac{2}{3}}$.
10. $9x^{2a}+15y^{2b}-15y^{-2b}-25x^{-2a}$.
11. $a^{2x}-a^{x}-\frac{7}{4}+a^{-x}+a^{-2x}$.
12. $x^{\frac{2}{a}}+x^{-\frac{2}{a}}+x^{2}-2+2x^{1+\frac{1}{a}}-2x^{1-\frac{1}{a}}$.
13. $2a+2(a^2-b^2)^{\frac{1}{2}}$.
14. $a+b+(a-b)^{-1}-2(a+b)^{\frac{1}{2}}(a-b)^{-\frac{1}{2}}$.
15. $x^{\frac{1}{2}}-3a^{\frac{1}{2}}$.
16. $x+3x^{\frac{1}{2}}+9$.
17. $a^{x}+4$.
18. $x^{2a}-2x^{a}+4$.
19. $c^{x}+c^{-\frac{x}{2}}$.
20. $1+2a^{-1}+4a^{-2}$.
21. $a^{2x}-x^{3}$.
22. $x^{-3}-x^{-2}+x^{-1}-1$.
23. $x^{\frac{4}{3}}+x+x^{\frac{2}{3}}+x^{\frac{1}{3}}+1$.
24. $x^{4n}-2x^{3n}+4x^{2n}-8x^{n}+16$.
25. $x^2+2x^{\frac{3}{2}}+x-16$.
26. $4x^{\frac{2}{3}}+16x^{\frac{1}{3}}+16-9x^{-\frac{2}{3}}$.
27. $4-x^{\frac{2}{3}}+4x+x^2$.
28. $a^{2x}-49-42a^{-x}-9a^{-2x}$.
29. $a^{\frac{1}{3}}(a^{\frac{1}{3}}-2b^{\frac{1}{3}})$.
30. 1.
31. $\dfrac{x^{\frac{2}{3}}-2}{x^{\frac{2}{3}}+2}$.
32. $\dfrac{a^{\frac{1}{2}}}{b}$.

XXXI. a. Page 255.

1. $\sqrt[12]{x^4}$.
2. $\dfrac{1}{\sqrt[12]{a^6}}$.
3. $\sqrt[12]{\dfrac{x}{a}}$.
4. $\sqrt[12]{a^9}$.
5. $\sqrt[12]{a^{21}}$.
6. $\sqrt[12]{a^4}$.
7. $\sqrt[n]{x^{\frac{2n}{3}}}$.
8. $\sqrt[n]{x^{an}}$.

9. $\sqrt[n]{a^{\frac{n}{2}}}$. **10.** $\dfrac{1}{\sqrt[n]{a^{\frac{1}{2}}}}$. **11.** $\sqrt[n]{x^{\frac{n^2}{3}}y^{\frac{1}{3}}}$. **12.** $\sqrt[n]{a^n}$.

13. $\dfrac{1}{\sqrt[n]{x^{\frac{n}{2}}y^{2n}}}$. **14.** $\sqrt[n]{a^{\frac{n}{2}}x^{n^2}}$. **15.** $\sqrt[18]{a^9}$, $\sqrt[18]{a^{10}}$. **16.** $\sqrt[10]{a^6}$, $\sqrt[10]{a^5}$.

17. $\sqrt[24]{x^9}$, $\sqrt[24]{x^{16}}$, $\sqrt[24]{x^6}$. **18.** $\sqrt[12]{x^3}$, $\sqrt[12]{x^{10}}$. **19.** $\sqrt[21]{a^3b^4}$, $\sqrt[21]{a^3b^3}$.

20. $\sqrt[26]{a^{13}x^{26}}$, $\sqrt[26]{a^6x^4}$. **21.** $\sqrt[6]{125}$, $\sqrt[6]{121}$, $\sqrt[6]{13}$.

22. $\sqrt[8]{64}$, $\sqrt[8]{81}$, $\sqrt[8]{6}$. **23.** $\sqrt[3]{2}$, $\sqrt[3]{2}$, $\sqrt[3]{2}$.

XXXI. b. Page 256.

1. $12\sqrt{2}$. **2.** $7\sqrt{3}$. **3.** $4\sqrt[3]{4}$. **4.** $6\sqrt[3]{2}$.

5. $15\sqrt{6}$. **6.** $24\sqrt{5}$. **7.** $35\sqrt{5}$. **8.** $7\sqrt[3]{3}$.

9. $5\sqrt[4]{5}$. **10.** $-9\sqrt[3]{3}$. **11.** $6a\sqrt{a}$. **12.** $3ab^2\sqrt{3ab}$.

13. $-3xy\sqrt[3]{4x}$. **14.** $x^3y^2\sqrt[n]{y^5}$. **15.** $xy^2\sqrt[p]{x^a}$. **16.** $(a+b)\sqrt{a}$.

17. $2(x-y)\sqrt[3]{xy}$. **18.** $\sqrt{242}$. **19.** $\sqrt{980}$.

20. $\sqrt[3]{864}$. **21.** $\sqrt[3]{750}$. **22.** $\sqrt{\dfrac{14}{11}}$. **23.** $\sqrt{5b}$.

24. $\sqrt{9a^2y}$. **25.** $\sqrt{\dfrac{3a}{x}}$. **26.** $\sqrt[3]{8ax}$. **27.** $\sqrt[4]{2a}$.

28. $\sqrt[n]{a^2b^2}$. **29.** $\sqrt[p]{ab}$. **30.** $\sqrt{\dfrac{x}{y}}$. **31.** $\sqrt{x^2-y^2}$.

32. $\sqrt{\dfrac{a+x}{a-x}}$. **33.** $14\sqrt{5}$. **34.** $\sqrt{7}$. **35.** $-12\sqrt{11}$.

36. $-15\sqrt{3}$. **37.** $7\sqrt[3]{7}$. **38.** $11\sqrt[3]{3}$. **39.** 0.

40. $17\sqrt[3]{2}$. **41.** $20\sqrt{3}-13\sqrt{2}$. **42.** $3\sqrt{6}$.

43. $6\sqrt{7}-15\sqrt{6}$. **44.** $\dfrac{181\sqrt{3}}{9}$.

XXXI. c. Page 259.

1. $14\sqrt{6}$. **2.** $12\sqrt{3}$. **3.** $10\sqrt{3a}$. **4.** $30\sqrt{3}$.

5. $288\sqrt{2}$. **6.** $\sqrt[3]{x^2-4}$. **7.** $3\sqrt{3}$. **8.** $\dfrac{5\sqrt{2}}{4}$.

9. $-\sqrt{13}$. **10.** $14\sqrt[3]{9}$. **11.** $240\sqrt[3]{4}$. **12.** $\sqrt{6}$.

13. $ab^3\sqrt{ab}$. **14.** $\dfrac{33}{10}$. **15.** $\dfrac{1}{10}\sqrt{2}$. **16.** $\dfrac{2\sqrt{2}}{a}$.

17. $\frac{a-b}{x}$. **18.** 9·8995. **19.** 11·1804. **20.** 3·7796.

21. 19·5959. **22.** 26·8328. **23.** 58·7878. **24.** ·8165.

25. ·2887. **26.** ·0447. **27.** ·2566. **28.** 1·5749.

29. ·4032.

XXXI. d. Page 260.

1. $6x-10\sqrt{x}$. **2.** $2x-2\sqrt{ax}$. **3.** $a\sqrt{b}+b\sqrt{a}$.

4. $x+y-\sqrt{x+y}$. **5.** $30+12\sqrt{6}$. **6.** $6\sqrt{21}-46$.

7. $6+\sqrt{10}$. **8.** $6a-6x+5\sqrt{ax}$. **9.** $x-1+\sqrt{x^2-x}$.

10. $x+a-\sqrt{x^2-a^2}$. **11.** $5a+x-4\sqrt{x^2+ax}$.

12. $1+8a-4\sqrt{a+4a^2}$. **13.** $2a-2\sqrt{a^2-x^2}$.

14. $a+x+2-3\sqrt{a+x}$. **15.** $2\sqrt{6}$. **16.** $16+6\sqrt{10}$.

17. $4x-2\sqrt{4x^2-a^2}$. **18.** $2x^2+2\sqrt{x^4-4y^4}$.

19. $2m+2\sqrt{m^2-n^2}$. **20.** $13a^2+5b^2-12\sqrt{a^4-b^4}$.

21. $63-18x\sqrt{14-4x^2}$. **22.** $8x^2-2\sqrt{16x^4-1}$.

XXXI. e. Page 262.

1. 113. **2.** -166. **3.** 172. **4.** -6.

5. $a-4b$. **6.** $9c^2-4x$. **7.** x. **8.** $2p-q$.

9. $2x$. **10.** $25(x^2-3y^2)-49a^2$. **11.** $\frac{11-3\sqrt{7}}{2}$.

12. $\frac{3\sqrt{7}-2\sqrt{3}}{3}$. **13.** $\frac{19-6\sqrt{2}}{17}$. **14.** $2+\sqrt{6}$.

15. $\frac{\sqrt{xy}}{y}$. **16.** $\frac{\sqrt{5}}{5}$. **17.** $\frac{\sqrt{ax}}{a-x}$. **18.** $4+\sqrt{15}$.

19. $5+\sqrt{6}$. **20.** $8-\sqrt{42}$. **21.** $\frac{\sqrt{7}-\sqrt{2}}{5}$.

22. $3\sqrt{2}-2\sqrt{3}$. **23.** $x-\sqrt{x^2-y^2}$. **24.** $\sqrt{x^2+a^2}-a$.

25. $\frac{1-\sqrt{1-x^4}}{x^2}$. **26.** $\frac{7a+b+8\sqrt{a^2-b^2}}{3a+5b}$.

27. $\frac{18+x^2-6\sqrt{9+x^2}}{x^2}$. **28.** $\sqrt{3}$.

29. $2-\sqrt{3}=$ ·26795. **30.** $11+5\sqrt{5}=$ 22·18035.

31. $\sqrt{5}-\sqrt{3}=$ ·50402. **32.** $\sqrt{5}+2=$ 4·23607.

33. $\frac{\sqrt{5}}{2}=$ 1·11803. **34.** $\frac{3\sqrt{3}-5}{2}=$ ·09807.

XXXI. f. Page 266.

1.	$\sqrt{5}-\sqrt{2}$.	2.	$\sqrt{10}+\sqrt{3}$.	3.	$\sqrt{7}-1$.
4.	$\sqrt{3}+\sqrt{2}$.	5.	$3\sqrt{7}+2\sqrt{3}$.	6.	$\sqrt{10}-2\sqrt{2}$.
7.	$4\sqrt{2}-3$.	8.	$2\sqrt{5}+3\sqrt{7}$.	9.	$2\sqrt{11}-\sqrt{3}$.
10.	$\frac{1}{2}\sqrt{5}+1$.	11.	$2-\frac{1}{3}\sqrt{3}$.	12.	$5\sqrt{\frac{1}{2}}+\sqrt{\frac{7}{2}}$.
13.	$\sqrt[4]{3}(\sqrt{2}+1)$.	14.	$\sqrt[4]{2}(\sqrt{3}-1)$.	15.	$\sqrt[4]{5}(\sqrt{2}+1)$.
16.	$\sqrt{2}+1$.	17.	$\sqrt{5}+1$.	18.	$\frac{1}{2}(\sqrt{5}+1)$.
19.	$\frac{1}{\sqrt[4]{2}}(\sqrt{3}+1)$.	20.	$\sqrt{3}-\sqrt{2}$.	21.	$\sqrt[4]{2}(\sqrt{5}+\sqrt{3})$.
22.	$\sqrt{2}-1$.	23.	$\sqrt{3}+1$.	24.	$\sqrt{5}-1$.
25.	$4+\sqrt{3}$.	26.	$\sqrt{5}+\sqrt{3}$.	27.	$\sqrt{7}-\sqrt{2}$.
28.	$2\sqrt{2}+\sqrt{3}$.	29.	$2\sqrt{2}-\sqrt{7}$.	30.	$\sqrt{11}+3\sqrt{2}$.

XXXI. g. Page 268.

1.	14.	2.	33.	3.	20.	4.	44.	5.	13.
6.	$\frac{6}{5}$.	7.	$\frac{17}{6}$.	8.	9.	9.	7.	10.	$\frac{56}{5}$.
11.	144.	12.	2.	13.	$\frac{121}{25}$.	14.	$\frac{25}{16}$.	15.	5.
16.	12.	17.	1.	18.	9.	19.	8.	20.	12.
21.	$\frac{1}{51}$.	22.	1.	23.	2.	24.	$(b-a)^2$.	25.	$\frac{(a-b)^2}{2a-b}$.
26.	$0, a-b$.	27.	10.	28.	$\frac{5}{2}$.	29.	2.	30.	± 1.

XXXI. h. Page 269.

1.	49.	2.	4.	3.	49.	4.	$\frac{121}{9}$.	5.	16.
6.	64.	7.	$\frac{64}{9}$.	8.	$\frac{1}{3}$.	9.	$\frac{4}{3}$.	10.	9.
11.	4.	12.	1.	13.	4.	14.	50.	15.	11.
16.	3.	17.	6.	18.	25.	19.	$\frac{1}{4}$.	20.	$\frac{8}{5}$.
21.	6.	22.	361.						

XXXI. i. Page 270.

1. $2a$. 2. 16. 3. $\dfrac{2ac}{c^2+1}$. 4. 0 or 36. 5. $\dfrac{1}{2}$.

6. $\pm\dfrac{a}{\sqrt{a^2-4}}$. 7. $\pm\dfrac{\sqrt{5}}{2}$. 8. ± 2. 9. $\pm\dfrac{5\sqrt{3}}{\sqrt{7}}$. 10. $\pm\sqrt{3}$.

11. 8. 12. $\sqrt{ab}$. 13. 7. 14. 5. 15. $-\dfrac{5}{4}$.

XXXII. Page 276.

1. Rational. 2. Rational. 3. Equal, but opposite in sign.
4. Imaginary. 5. Imaginary. 6. Equal, but opposite in sign.
7. $x^2-2x-15=0$. 8. $x^2+20x+99=0$.
9. $x^2-2ax+a^2-b^2=0$. 10. $12x^2-28x+15=0$.
11. $15x^2+2ax-8a^2=0$. 12. $8x^2-7x=0$. 13. $-\frac{1}{2}$.
15. Sum $\dfrac{4}{3}$, difference $\dfrac{2\sqrt{7}}{3}$, sum of squares $\dfrac{22}{9}$. 17. $x^2-6x+4=0$.
18. $x^2+4x+1=0$. 19. $30x^2+(6a-5b)x-ab=0$. 20. $4x^2-16x+9=0$.
21. $(a^2-b^2)x^2-2(a^2+b^2)x+a^2-b^2=0$. 22. $4abx^2-2(a^2+b^2)x+ab=0$.
23. $\dfrac{q^2-2pr}{p^2}$. 24. $\dfrac{q^2-4pr}{p^2}$. 25. $-\dfrac{qr}{p^2}$. 26. $\dfrac{q^4-4prq^2+2p^2r^2}{p^4}$.
27. $\dfrac{qr^2(3pr-q^2)}{p^5}$ 28. $\dfrac{q(3pr-q^2)}{p^2r}$. 29. $P=p(p^2-3q)$, $Q=q^3$.
30. $b^2x^2-(a^3-3ab)x+b=0$. 31. $2b^2=9ac$.
32. $8x^2-20a^3x-a^6=0$. 36. 1·732. 37. $a-2b+3c$.
38. (i) $6x^2-37x+6=0$; (ii) $6x^2-35x+49=0$; (iii) $6x^2-49x+49=0$; (iv) $6x^2-13x+6=0$; (v) $6x^2-35x+25=0$.
39. (i) $k=8$ or 0, the roots are 6 or 2; (ii) $k=\pm 2$, the roots are ∓ 3; (iii) $k=\pm\frac{5}{4}$, $-\frac{5}{4}$.
40. (i) $x^2-12x+32=0$; (ii) $x^2-4x=0$. 41. $x^2-10x+19=0$.
42. $x^2-x-14=0$. 43. $ax^2-(2al-b)x+(al^2-bl+c)=0$.
45. (i) $3p^2=16q$; (ii) $4p^2=25q$; (iii) $mp^2=(m+1)^2q$.
47. (i) $qx^2-px+1=0$; (ii) $x^2-(p+q)x+pq=0$;
(iii) $q^2x^2-(p^2-2q)x+1=0$; (iv) $qx^2-(p^2-2q)x+q=0$;
(v) $qx^2-p(p^2-3q)x+q^2=0$;
(vi) $x^2-(2m+p)x+(m^2+pm+q)=0$;
(vii) $px^2-(p^2+q)x+pq=0$;
(viii) $x^2-2(p^2-2q)x+p^2(p^2-4q)=0$.
48. (i) 30; (ii) $7a^2$. 50. 15 or 13; 2 or $\frac{3}{2}$. 51. 4.

XXXIII. a. Page 280.

1. $2, -3, \frac{-1\pm\sqrt{-27}}{2}$. 2. $2, -\frac{1}{2}, \frac{1}{3}(1\pm\sqrt{10})$.

3. $\frac{-1\pm\sqrt{-3}}{2}, \frac{5\pm\sqrt{21}}{2}$. 4. $-2, -\frac{1}{2}$.

5. $16, -\frac{4}{3}$. 6. $3, 0$. 7. $2, \frac{15}{4}$.

8. $9, \frac{1}{9}$. 9. $3, -2, 1, -6$. 10. $1, \frac{1}{81}$.

11. $20, 11$. 12. $2, -1$. 13. $-8, -1, 0$.

14. 0. 15. $2, \frac{3}{2}, \frac{7\pm\sqrt{33}}{4}$. 16. $1, 1\pm2\sqrt{15}$.

17. $7, -1, 3\pm2\sqrt{2}$. 18. $3, -\frac{7}{2}, \frac{-3\pm\sqrt{1357}}{12}$.

19. $\frac{3\pm\sqrt{5}}{2}, \frac{9\pm\sqrt{-83}}{6}$. 20. $2, -\frac{1}{2}, \frac{3\pm\sqrt{505}}{4}$. 21. $\frac{2}{13}, \frac{8}{7}$.

22. $a^3, \frac{1}{a}$. 23. $1, \frac{c-a}{a-b}$. 24. $1, \frac{c(a-b)}{a(b-c)}$.

25. a, b. 26. $c, \frac{a^2+b^2-ac-bc}{a+b-2c}$. 27. $p+q$.

28. $8, -2$. 29. $\frac{5}{2}, -\frac{13}{2}$. 30. a, b.

31. 10 or -25. 32. 4 or -1. 33. 12 or 10.

34. $3, -5, -1\pm\sqrt{-1}$. 35. $1, -3, -\frac{1}{2}, -\frac{3}{2}$.

XXXIII. b. Page 282.

1. $x=5, \quad y=2$. 2. $x=4, 3;\ y=3, 4$.
3. $x=7, -2;\ y=2, -7$. 4. $x=8, 2;\ y=2, 8$.
5. $x=12, 3;\ y=3, 12$. 6. $x=3, \quad y=6$.
7. $x=\frac{1}{7}, y=\frac{1}{3}$. 8. $x=4, 1, \frac{1}{2}(-5\pm\sqrt{41});\ y=1, 4, \frac{1}{2}(-5\mp\sqrt{41})$.
9. $x=\pm10, 0;\ y=\pm1, \pm\frac{\sqrt{34}}{2}$. 10. $x=7, -\frac{35}{4};\ y=3, -\frac{15}{4}$.
11. $x=5, 2, 1\pm\sqrt{6};\ y=-2, -5, -1\pm\sqrt{6}$.

12. $x=\frac{3}{4}$, 1, 0; $y=\frac{3}{4}$, 0, 1.

13. $x=-1$, $5\pm\sqrt{6}$; $y=-1$, $1\pm\sqrt{\frac{2}{3}}$. [It may be shewn that $(x+1)^3=27(y+1)^3$.]

14. $x=4$ or $-\frac{8}{3}$, $y=1$ or $-\frac{2}{3}$; or $x=\frac{2\pm2\sqrt{15}}{7}$, $y=\frac{-2\mp2\sqrt{15}}{7}$.

15. $x=16$, 1; $y=1$, 16. **16.** $x=17$, $y=\pm8$. **17.** $x=2$, $y=\frac{1}{2}$.

XXXIV. a. Page 286.

1. $6:1$. **2.** $\frac{1}{5}$. **3.** $4:1$. **4.** $17:7$. **5.** $3:4$.

6. $5:4$. **7.** 21, 28. **8.** 11. **9.** 27.

XXXIV. b. Page 289.

1. bc. **2.** $\frac{6b^3}{a}$. **3.** $5y^2$. **4.** b.

5. $4x$. **6.** $12xy^2$. **7.** x^2. **8.** ab.

9. $4x^2$. **10.** $6a^2x$. **11.** $9ab^2$. **17.** 8 or $\frac{2}{3}$.

18. $x=17$, $y=11$. **19.** 2 or 0. **20.** 5 or 0.

Miscellaneous Examples V. Page 290.

1. $\frac{c^{\frac{1}{12}}}{a^{\frac{1}{3}}b^{\frac{1}{4}}}$; 1. **5.** $4\sqrt{2}$. **8.** (1) $x^{2b}+x^{-2b}$. (2) $(a+b)^2$.

9. $-\frac{32}{7a}$. **10.** (1) $\frac{b^4}{a^2}$. (2) $\frac{7}{8}$. **11.** 1. **13.** $a+b$.

15. $q=4$. The other root is 1. **16.** $1\frac{5}{6}$ hours. **19.** -1.

20. (1) 2, 3, $\frac{5\pm\sqrt{37}}{2}$; (2) ±1, $\pm\sqrt{-3\pm2\sqrt{2}}$.

22. $4\frac{2}{3}$; 1·412. **23.** $x^2-2(a+b)x+2ab=0$.

24. (1) a, $\frac{a(a-b)}{b-c}$ [one root is evidently a, and the product of the roots is $\frac{a^2(a-b)}{b-c}$]; (2) q, $p-q$.

XXXV. a. Page 296.

7. 36. **8.** 32. **9.** 25. **11.** 1·2 sq. cm.

12. $y=3x$. Any point whose ordinate is equal to three times its abscissa.

14. (i) (8, 5); (ii) (10, 10).

15. (i) (4, 5); (ii) (4, 5); (iii) $(-4, -5)$; (iv) $(-4, -5)$.

16. (i) 17; (ii) 17; (iii) 2·5″; (iv) 2·5″.

17. (i) and (ii) 5; (iii) and (iv) 17; (v) and (vi) 37.

19. The lines are $x=5$, $y=8$. The point (5, 8).

20. 10. **23.** 68 units. **24.** 10, 13, 5, 5, 3.

25. A circle of radius 13 whose centre is at the origin.

27. 10, 12, 16, 6, 0. At the points (0, 10), $(-5, 0)$.

XXXV. b. Page 302.

21. 32 units of area. **22.** 1 sq. in.

23. 72 units of area. **24.** 0·64 sq. cm.

XXXV. c. Page 307.

1. $x=1$, $y=5$. **2.** $x=2$, $y=10$. **3.** $x=3$, $y=12$.

4. $x=3$, $y=-2$. **5.** $x=4$, $y=2$. **6.** $x=6$, $y=8$.

7. $x=-2$, $y=4$. **8.** $x=0$, $y=-3$. **9.** $x=-3$, $y=0$.

10. $x=-4$, $y=5$. **11.** At the point (0, 21). **12.** $3x+4y=7$.

XXXV. d. Page 309.

2. $x=1$, $y=18$. **3.** 9; 2·4.

4. 5 in each case. 18·5 units. **5.** 2·5, 1·7.

6. (3, 4), (4, 1), $(-3, 2)$. **7.** 7·5 sq. in.

8. 2·60, 5·63, 4·16, 5·77. **10.** (i) 53·7 grains; (ii) 0·2.

11. 39·3; 91·6; $y=0{\cdot}393x$. **12.** 90; 72. **13.** \$1.17, 125.

14. 112; 168; 78. $y=\frac{10}{11}x-70$.

15. $y=500+\frac{x}{2}$, \$1750, 4500.

XXXV. e. Page 318.

5. $11y=3x+35$. 16; 25. **6.** $y=0{\cdot}21x+1{\cdot}37$.

7. $y=0{\cdot}4x+1{\cdot}6$. 9·2; 3. **8.** 45·96; 39·40.

9. 3·85 in.; 17·6 in. **10.** 54·5° F.; 86·9° F.; $F=32+\frac{9}{5}C$.

11. 8·1 in.; 24·375 oz. **12.** 8·6; $P=0·14W+0·2$; 225 lbs.

13. 2·49 sq. ft.

14. 6 p.m., 48 mi. from Toronto. At 4 and 8 p.m.
(i) B 4 mi. behind A; C 6 mi. behind B. (ii) 4.21 p.m.

15. 4.30 p.m., 18 mi. from O. (i) At 3 and 6 p.m. (ii) 20 mi.

16. (i) 1 p.m., 28 mi. from P; (ii) 20 mi.; (iii) 11.30 am.

XXXV. f. Page 327.

4. $y=x$. **5.** (0, 0), (−4, 2).

6. (i) (0, 0); (ii) (4, 4); (iii) (−2, −3). **8.** (2, 1).

9. (i) 1·46, −5·46; (ii) 3·24, −1·24; (iii) 3·32, 0·68;
(iv) 4, −8; (v) 4, −2; (vi) 1·5, 2·5.

10. 2·38, 4·62; −1·25. **11.** −5; 7.

12. −0·25; 3·79, −0·79; 4·62, −1·62.

XXXV. g. Page 334.

1. (i) $x=2$, or -7; $y=7$, or -2. (ii) $x=8$, or 6; $y=6$, or 8.
(iii) $x=3$, or $-5·8$; $y=5$, or $-0·6$. (iv) $x=5·2$, or $-1·3$;
$y=-2·9$, or 5·7.

2. The straight line $3x+4y=25$ *touches* the circle $x^2+y^2=25$ at the point (3, 4).

3. (i) 1·46, −5·46; (ii) 3·24, −1·24; (iii) 3·32, 0·68.

4. 6·46, −0·46. 12. **6.** $3·3\dot{0}$, −0·30.

9. −5 and 1. **10.** 1·5. **11.** −0·25.

12. (i) $x=12$, or 3; $y=3$, or 12; (ii) $x=6$, or -3; $y=-3$, or -6.
(iii) $x=2, 3, -3, -2$;
$y=3, 2, -2, -3$.

XXXV. h. Page 340.

1. 6 p.m.; (i) 3.30 p.m.; (ii) 7 p.m. **2.** (i) 2 p.m.; 2.52 p.m.

3. 47 mi. from A's starting place at 12.42 p.m.
11.12 a.m. and 2.12 p.m.

4. 40 yds. A 16 yds. ahead, C 16 yds. behind.

5. 5 secs. **6.** 5 hours from the start.

7. 7.36 p.m.; 3 p.m. and 5 p.m.; 19 mi. from Y.

8. 12.12 p.m. (i) 11 a.m.; (ii) 57 mi. **9.** 5 mi. **10.** 400 yds.

11. 0·52, 2·9, 11·6; 2·75, 2·3, 3·1. **14.** 2·080, 2·140.

Miscellaneous Examples VI. Page 342.

1. 0. **2.** $-6a-2b-4d$. **3.** $\frac{1}{2}x^2+\frac{1}{2}xy-\frac{2}{9}y^2$.

4. 3. **5.** $4-12x+13x^2-6x^3+x^4$. **6.** $4\frac{1}{3}$.

7. $a-2$. **8.** $\frac{a^2+b^2}{a^2-b^2}$. **9.** $x=15,\ y=16$.

10. 1, 3. **11.** $\frac{4}{9}$. **12.** $\frac{13}{12}x^2-\frac{11}{20}y^2+\frac{5}{3}z^2$.

13. $x^3+24x^2y+192xy^2+512y^3$. **14.** x^2-y^2. **15.** 11.

16. H.C.F. $(x+2)(x-1)$. L.C.M. $(x-1)(x+2)^2(x^2+2)$.

17. $2a^2-3a+3$. **18.** $x=\frac{8b+7a}{9},\ y=\frac{8a+7b}{9}$. **19.** $\frac{x}{a-x}$.

20. $\frac{5}{3}$. **22.** $x-\frac{3}{4}y$. **23.** $-35x+18y+17z$.

24. (1) $(10x-1)(x+8)$.
(2) $(3x-y)(3x+y)(9x^2+3xy+y^2)(9x^2-3xy+y^2)$.

25. 13. **26.** 2. **27.** x^2-1.

28. $\frac{x-3y}{x+3y}$. **29.** $x=\frac{2}{a},\ y=3b$.

30. $21\frac{9}{11}'$ and $54\frac{6}{11}'$ past 7. **31.** 1.

32. $\frac{1}{8}x^6-8y^6$. **33.** $2a^4+12a^2+2$. **34.** 884.

35. $\frac{1}{x^2-y^2}$. **36.** $x=14,\ y=17$. **37.** $2x^3-3x+7$.

38. $x=\frac{2}{3}$. **39.** $x(x^3+y^3)(3x-y)$.

40. 10 quarters, 20 half-dollars. **41.** $8ab$.

42. $2ab^3+3b^4$. **43.** 36. **44.** $4x-5$.

45. $\frac{a^2+b^2}{ab(a-b)^2}$. **46.** $2x-\frac{y}{6}$.

47. $x=\frac{1}{2},\ y=\frac{1}{3}$; or $x=0,\ y=0$. **48.** $\frac{2(x-7)(2x-7)}{(x-2)(x-3)(x-4)(x-5)}$.

49. $(2x+3)(4x+5)(3x-5)(x+2)(x-2)$. **50.** 75 cents.

51. $-5605x+5589$. **52.** $4a^2-9b^2+24bc-16c^2$.

54. $6(x+1)(x-3)(x-4)$. **55.** $2(a^2+b^2)(x^2+y^2)$.

56. $x=3,\ y=2,\ z=1$. **57.** 0. **58.** $x=-5$.

59. $\frac{x^4+2y^4}{y^2(x-y)^2(x^2+xy+y^2)}$. **60.** 24 days. **61.** $\frac{3}{2}x^3-5x^2+\frac{x}{4}+9$.

62. 94. **63.** $\frac{1}{x^2-1}$. **64.** $\frac{ac}{b}x^2 - \frac{b}{c}x$.

65. $x=2\frac{1}{2}$. **66.** (1) $(x^2+1)(x+5)$. (2) $(x-19y)(x+17y)$.

67. $x=24, y=9, z=5$. **68.** $\frac{2x}{x+5y}$. **69.** $(2a-3b+2c)^2$.

70. 3. **71.** $x^2+y^2+z^2$. **72.** $-2ab$.

73. 6. **74.** (1) $3x(x+9)(x-7)$. (2) $(a+b+1)(a+b)$.

75. $x=\frac{qr}{p^2+q^2}$, $y=\frac{pr}{p^2+q^2}$. **76.** $\frac{2x^2}{8x^3-y^3}$.

77. $x=8$. **78.** $\frac{x^3+x^2-2}{2x^2+2x+1}$. **79.** $\frac{2}{(1-x^2)^2}$.

80. 640. **81.** $1-4x-\frac{46}{15}x^2+\frac{10}{3}x^3$.

82. $\frac{1}{2}$. **83.** $\frac{3}{4}x^5-4x^4+\frac{77}{8}x^3-\frac{43}{4}x^2-\frac{33}{4}x+27$.

84. $2a^2b^2(a-2b)^2(2a+b)^2$. **85.** $x=5$.

86. $\frac{5x^2-4x-8}{3x^2+4x+24}$. **87.** $2a^2+3a+\frac{3}{a}$. **88.** $x=2ab$, $y=3ab$.

89. $3(2x-y)(5x+4y)$. **90.** 25 ten-cent pieces, 30 quarters. **91.** 0.

92. $x^3-x^2+\frac{5}{3}x-\frac{23}{9}$. Rem. $-\frac{163}{9}$. **93.** $60(p^6-q^6)$.

94. (1) $(a-2b^5)(a^2+2ab^5+4b^{10})$. (2) $(x^2+x-1)(x^2-x+1)$.

95. $x=a-2b$. **96.** (1) $\frac{7bc}{13a^3}$. (2) $\frac{y(y^2-y+2)}{2(y+5)}$.

97. $x=1$, $y=-1$, $z=0$. **98.** $\frac{(y+1)(y-5)}{(y-1)(y-6)}$.

99. $\frac{2a-3b+c}{2a-3c}$. **100.** Twelve minutes past four.

101. (1) $5\frac{1}{6}$. (2) -1. **102.** (1) $\frac{2(a+x)}{a^2+ax+x^2}$. (2) $1+x-x^3$.

103. $a^3-3a+\frac{3}{a}-\frac{1}{a^3}$; $a-\frac{1}{a}$. **104.** $1-5x+15x^2-45x^3$.

105. \$300. **106.** $(2a-3b)(a+b)$.

107. (1) 2 or $\frac{7}{5}$. (2) $\frac{5}{2}$ or $\frac{4}{3}$. **109.** $7x^2-\frac{x}{5}+3$.

110. $\frac{3}{2a^2}$. **111.** $\frac{3x^2+7x-12}{(x^2-9)(x^2-16)}$; $\frac{1}{(x-3)(x-4)}$.

112. H.C.F. $x-5b$. L.C.M. $6(x+3a)(x-3a)(x-5b)$.

113. (1) $\frac{ab}{2c}$ or $2d$. (2) 9 or -3. **114.** 177. **115.** 18 miles.

116. $(3x+2y)^2+(3x+2y)(2x+3y)+(2x+3y)^2=19x^2+37xy+19y^2$.

118. (1) $(x+y)(x+y)(x+y)$. (2) $mn(m-n)$.

119. (1) $\left.\begin{matrix} x=\ \ \ 3, & 1 \\ y=-1, & -3 \end{matrix}\right\}$. (2) $\left.\begin{matrix} x=7, & -5 \\ y=2, & -2 \end{matrix}\right\}$. **120.** a^2+b^2.

121. (1) 1. (2) $\frac{x-1}{x^2}$. **122.** $a-b$. **123.** 435.

124. (1) $x=a\pm b$. (2) $x=3,\ y=2$.

125. (1) $\frac{x(x+y+z)}{z(x-y+z)}$. (2) $\frac{3x^2+1}{4x(x^2+1)}$. **126.** $x^2+(a+2)x+3$.

127. (1) $(x-3y)(x+8y+1)$. (2) $x\left(x+\frac{2}{x}\right)\left(x-\frac{2}{x}\right)$.

128. $p-\frac{3q}{2p}-\frac{9q^2}{8p^3}$. **129.** (1) $x=4\frac{1}{2}$. (2) $x=\frac{b}{a^2-ab+b^2}$, $y=\frac{a}{a^2-ab+b^2}$.

130. $x+a$. **131.** (1) x^{a+b+c}. (2) $x^{\frac{5}{12}}y^{\frac{13}{12}}$.

132. \$1. **134.** $x-4+\frac{2}{x}$.

135. H.C.F. x^2+a^2. L.C.M. $(x^2+a^2)(x^2-4a^2)$.

136. (1) $-2y$. (2) 3. **137.** $(x-2a)(x^2+2ax+4a^2)(2a+3b)(2a-3b)$.

138. (1) 3. (2) $x=105$, $y=210$, $z=420$. **139.** The difference is 3.

140. $x^3-4y^3+9z^3-12y^{\frac{3}{2}}z^{\frac{3}{2}}$. **141.** 3 hrs. 36 min.

142. (1) $\frac{3}{2}$. (2) $\frac{8}{9}$. **143.** $\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}-1$.

144. (1) $\frac{1}{x^3+1}$. (2) $\frac{a^2+y^2}{a-y}$. **145.** (1) $4(\sqrt{2}+\sqrt{3})$. (2) $\sqrt{21}+\sqrt{14}$.

146. (1) $x=\frac{a^2+b^2}{a+b}$. (2) $x=2\frac{1}{2}$ or $-1\frac{3}{4}$, $y=-1\frac{1}{6}$ or $1\frac{2}{3}$.

148. $a^{\frac{1}{2}}-2a^{\frac{1}{4}}x^{\frac{1}{4}}+x^{\frac{1}{2}}$. **149.** a^2+1. **150.** \$3.

152. 0. **153.** $\frac{x^2}{(7x+4)(4x-3)}$.

155. (1) $x=7$ or $-\frac{77}{2}$. (2) $x=\pm5$ or $\pm2\sqrt{3}$, $y=\pm3$ or $\pm\frac{\sqrt{3}}{3}$.

156. (1) 0. (2) $x^{\frac{7}{12}}y^{\frac{5}{6}}$. **157.** $(p+1)x-(p-1)$. **158.** $\frac{3a}{5b(x^2y^2-1)}$.

159. (1) $1-2^{2n}$. (2) 3^n-2^n. **160.** 5 hrs. $57', 47\frac{1}{2}''$.

161. 12. **162.** (1) $\left(\frac{ab}{2a+b}\right)^2$. (2) $\frac{3}{5}$.

163. 1. **164.** 1, $5+\sqrt{7}$.

165. $(3y+2x)(3y-2x)(x+2)(x^2-2x+4)(x-2)(x^2+2x+4)$.

166. (1) 1. (2) $\frac{32}{3}$. **167.** (1) 1. (2) $\sqrt{11}-\sqrt{3}$.

168. H.C.F. $5x^2-1$. L.C.M. $=(5x^2-1)^2(4x^2+1)(5x^2+x+1)$.

169. (1) $\frac{(a-b)^2}{2b}$. (2) $-2\frac{5}{8}$, $-8\frac{5}{8}$. **170.** \$3 and \$6 a dozen.

172. 0. **173.** (1) $\frac{x^3+y^3+z^3}{3xyz}$. (2) 1.

174. 11. **175.** (1) n. (2) $2-\sqrt{3}$.

176. (1) $x=\frac{69}{20}a$; (2) $x=\pm\frac{1}{2}, \pm\frac{3}{2}$; $y=\pm\frac{3}{2}, \pm\frac{1}{2}$.

177. $\frac{ax+b^2x^2}{a^m+x^n}$. **178.** (1) $c^{\frac{13}{12}}$. (2) 27. **179.** 4 miles an hour.

181. $\frac{ac-b^2}{a+c-2b}$. **182.** (1) x^2+y^2+xy-1. (2) $\frac{1}{2x-1}$.

183. $3-2x^2$. **184.** (1) $2x$. (2) 10.

185. (1) 5, 1. (2) $x=6, 2, 4$; $y=2, 6, 4$. **186.** 0.

187. (1) $20\frac{3}{4}$ or $16\frac{3}{4}$. (2) $x=3$ or $\frac{1}{2}$, $y=-1$ or $\frac{2}{3}$.

189. $(2a+1)x-a$. **190.** $2x-\frac{1}{2x}$. **191.** $(b+c)(c+a)(a+b)$.

192. (1) $\frac{3x-4}{(x+1)(x^3-1)}$. (2) $\frac{2a}{\sqrt{x+a}}$.

194. Began at $16\frac{4}{11}'$ past 3, and ended $27\frac{3}{11}'$ past 5; walked 2 hours, $10\frac{10}{11}$ minutes.

196. $\left(\frac{p}{q}\right)^{p+q}$. **197.** (1) 47. (2) b. **199.** 20. **200.** $9x^2+y^2$.

[illegible]

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[illegible]

[illegible]

[illegible]

Rule of Indices.
1 Subtract indices for division
2 Add " " multiplication
3 Raising to powers multiply
4 Extracting roots divide

$\sqrt{5} + \sqrt{3}$
$\sqrt{5} - \sqrt{3}$
$\sqrt{a} + \sqrt{b}$
$\sqrt{a} - \sqrt{b}$

real - { rational / irrational

imaginary ex $\sqrt{-7}$

answer to problem of
solve equation [illegible]
is found on the other page

Take the non repeaters from the whole number. Use your answer as a numerator and for denominator put down as many nines as there are repeaters and add as many naughts as there are non repeaters.

$$V = .0\dot{5}$$
$$V = .05555$$
$$10V = .5555$$
$$100V = 5.5555$$

$$90V = 5$$
$$V = \frac{5}{90}$$

$$\frac{E(x)}{x-a} = Q + \frac{R}{x-a}$$

$\therefore E(x) = Q(x-a) + R$

Let $x = a$ then $x - a = 0$

$\therefore E(a) = R$

$\therefore$ the remainder on dividing $E(x)$ by $x - a$ is equal to a substituted for x in the original expression.

To find the condition on which the roots of an expression are reciprocals

Roots are reciprocals when the coef of x^n = abs. term

PRICE
50
CENTS